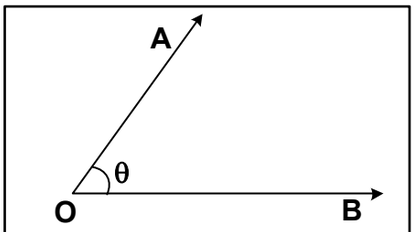
## ÁNGULO GEOMÉTRICO



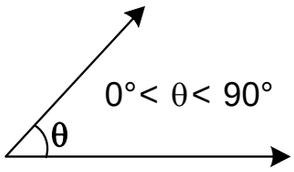

Notación: $\measuredangle AOB$

Medida: $m\measuredangle AOB = \theta$

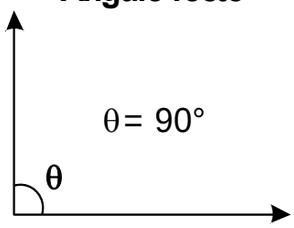
**Clasificación**

### Según la medida de sus ángulos

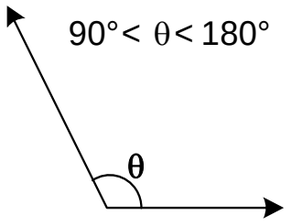
**Ángulo agudo**




**Ángulo recto**

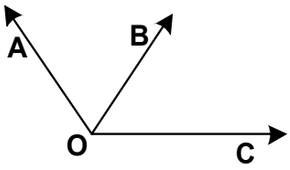



**Ángulo obtuso**

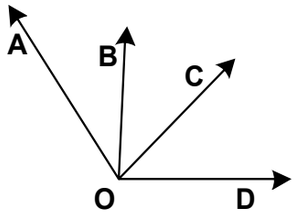



### Según la posición de sus lados

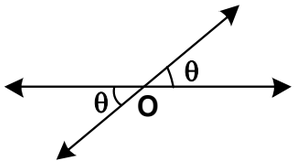
**Ángulos adyacentes (AOB y BOC)**




**Ángulos consecutivos (AOB, BOC y COD)**

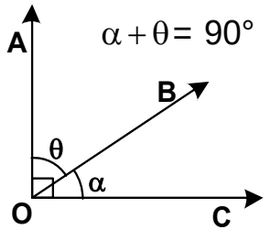



**Ángulos opuestos por un vértice**

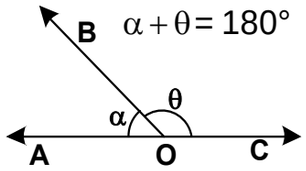



### Según la relación entre sus medidas

**Ángulos adyacentes complementarios**

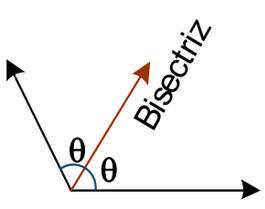


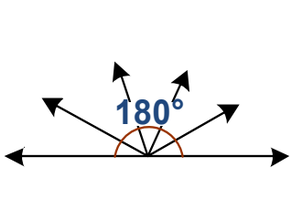

**Ángulos adyacentes suplementarios**

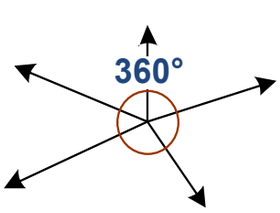



**Nota:**

$$m\measuredangle\theta \Rightarrow \begin{cases} C_\theta = 90° - \theta \\ S_\theta = 180° - \theta \end{cases}$$




**N° PAR de veces**

CCC ... $C_\theta = \theta$

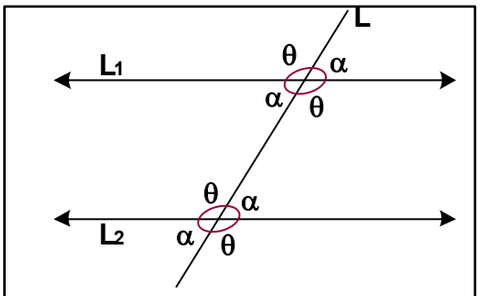
SSS ... $S_\theta = \theta$

**N° IMPAR de veces**

CCC ... $C_\theta = C_\theta$

SSS ... $S_\theta = S_\theta$

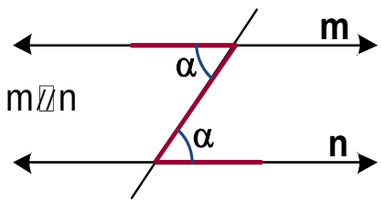
## ÁNGULO ENTRE RECTAS PARALELAS Y UNA SECANTE



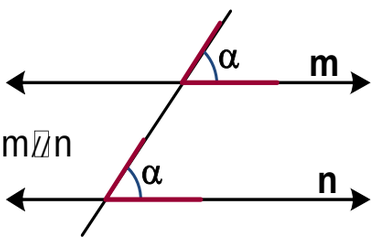

$L_1$ y $L_2$ rectas paralelas y
L recta secante.
Notación: $L1 // L2$

### Según su posición y su relación entre sus medidas, tenemos.

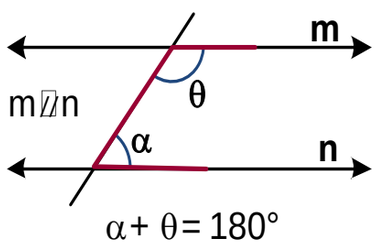
#### Ángulos alternos internos




#### Ángulos correspondientes

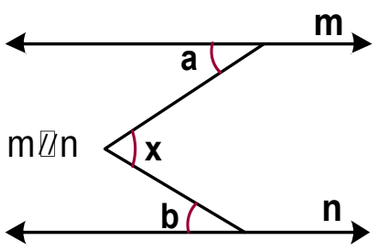



#### Ángulos conjugados internos

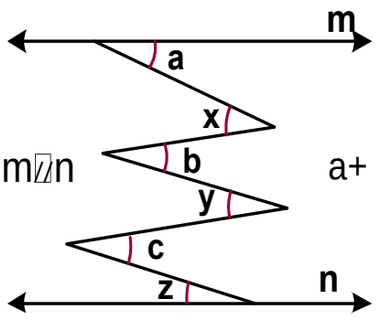



$\alpha + \theta = 180°$

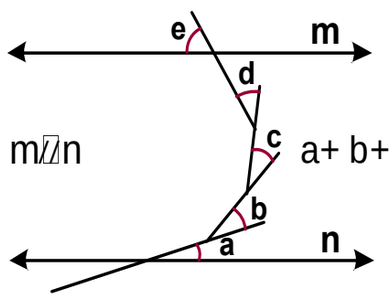
### Algunas propiedades de ángulos entre paralelas de uso frecuente.




$x = a + b$

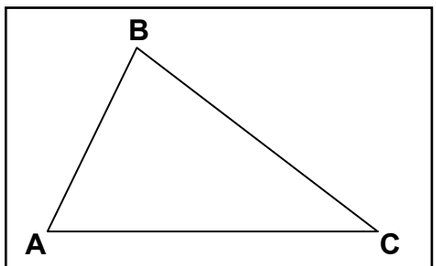



$a + b + c = x + y + z$




$a + b + c + d + e = 180°$

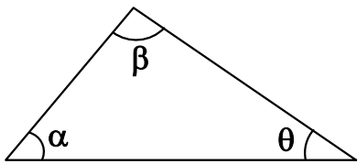
## TRIÁNGULOS




Vértices: A, B y C
Lados: $\overline{AB}$, $\overline{BC}$ y $\overline{AC}$
Notación: $\triangle ABC$

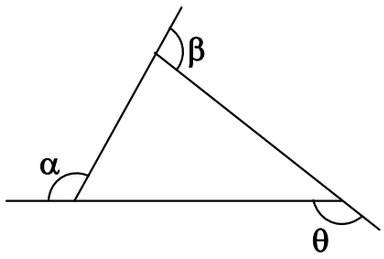
### Teoremas fundamentales

**La suma de medidas de ángulos interiores es 180°**



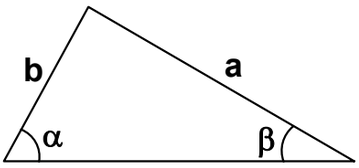

$\alpha + \beta + \theta = 180°$

**La suma de medidas de ángulos exteriores es 360°**




$\alpha + \beta + \theta = 360°$

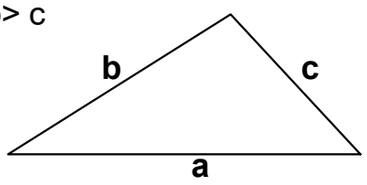
**En todo triangulo, al lado de mayor longitud, se le opone el ángulo de mayor medida, y viceversa.**



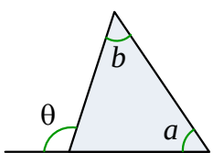
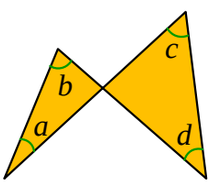
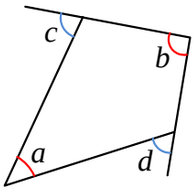
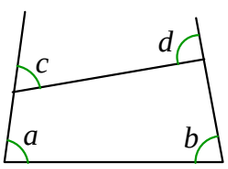
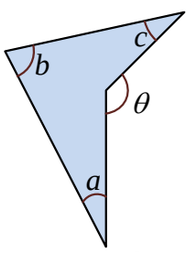

$a > b \Leftrightarrow \alpha > \beta$

**En todo triangulo, la longitud de un lado es menor que la suma de longitudes de los otros dos y mayor que la diferencia de las mismas.**

Si a> b> c




$b - c < a < b + c$

| Ratón | Mariposa | Pescadito | Carterita | Gumerand |
|---|---|---|---|---|
|  |  |  |  |  |
| $\theta = a + b$ | $a + b = c + d$ | $a + b = c + d$ | $a + b = c + d$ | $\theta = a + b + c$ |

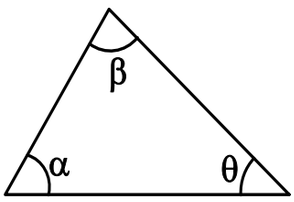
## CLASIFICACIÓN DE LOS TRIÁNGULOS

# CLASIFICACIÓN

## Según la medida de sus ángulos

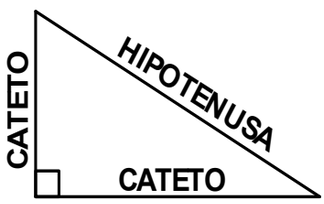
### Triángulo acutángulo

$\alpha$, $\beta$ y $\theta$ son agudos



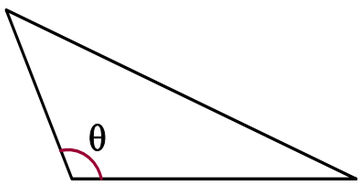

### Triángulo rectángulo

Un ángulo interior es recto




### Triángulo obtusángulo

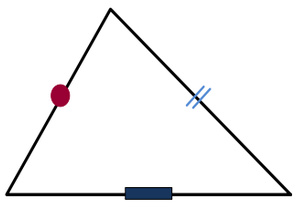
Un ángulo interior es obtuso




## Según la longitud de sus lados

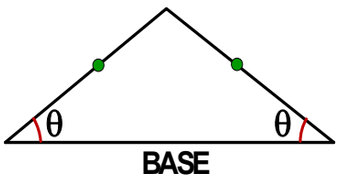
### Triángulo escaleno

Sus tres lados son diferentes



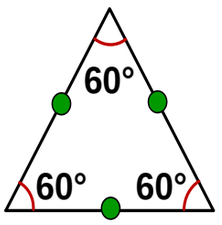
### Triángulo isósceles

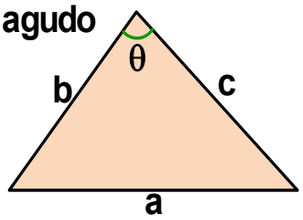
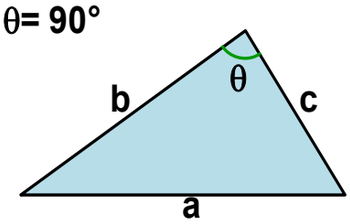
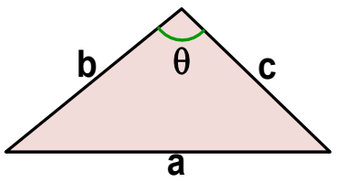
Dos de sus lados son iguales




### Triángulo equilátero

Sus tres lados son iguales



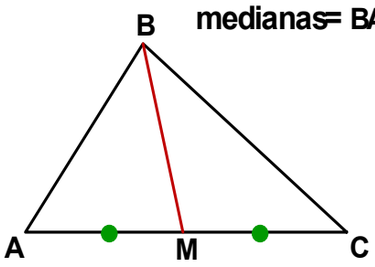

| $\theta$: agudo | $\theta$= 90° | $\theta$: obtuso |
|---|---|---|
|   |   |   |
| $a^2 < b^2 + c^2$ | $a^2 = b^2 + c^2$ | $a^2 > b^2 + c^2$ |

## LÍNEAS NOTABLES ASOCIADAS AL TRIÁNGULO

# LÍNEAS NOTABLES

### Mediana $\overline{BM}$

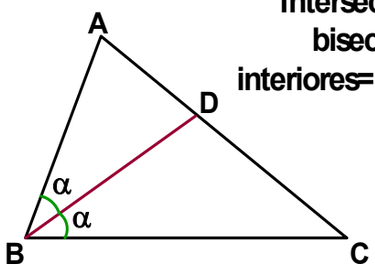
Intersección de medianas= BARICENTRO




### Bisectriz $\overline{BD}$

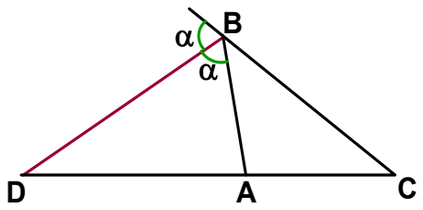
**Interior**

Intersección de bisectrices interiores= INCENTRO



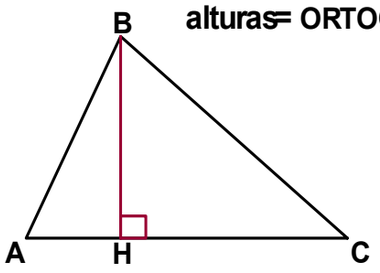

**Exterior**

Intersección de dos bisectrices exteriores= EXCENTRO

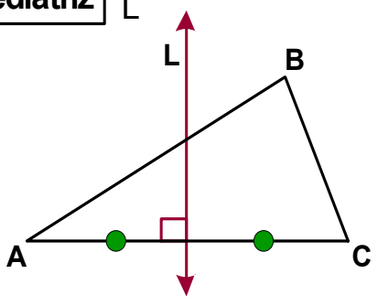



### Altura $\overline{BH}$

Intersección de alturas= ORTOCENTRO

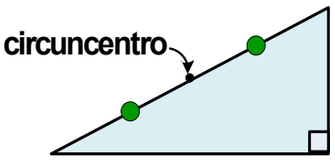


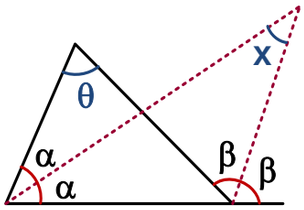

### Mediatriz $\overline{L}$



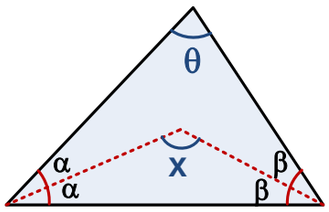

Intersección de mediatrices= CIRCUNCENTRO

**Nota:**

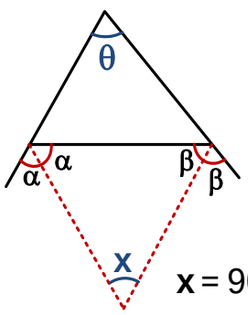



$$x = \frac{\theta}{2}$$

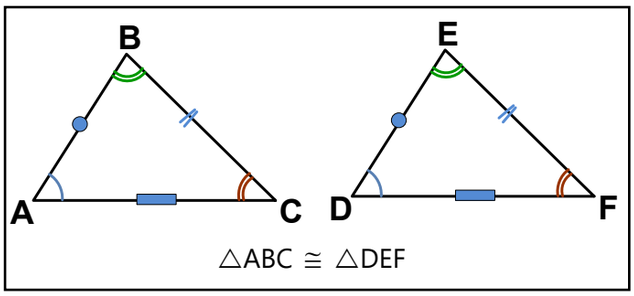



$$x = 90^\circ + \frac{\theta}{2}$$



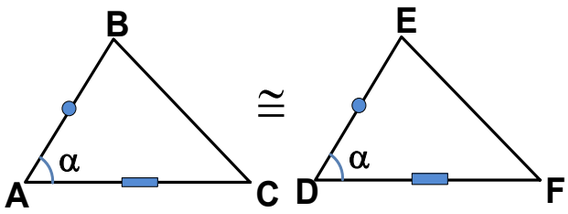

$$x = 90^\circ - \frac{\theta}{2}$$

## CONGRUENCIA DE TRIÁNGULOS

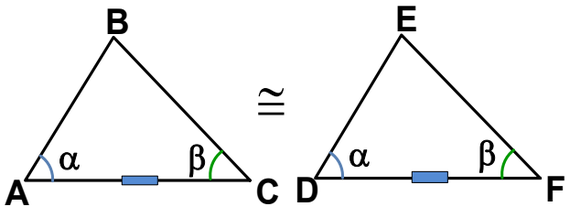



## Criterios de congruencia

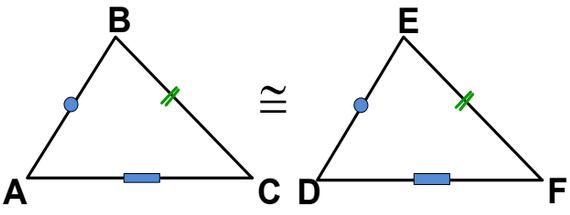
### Lado-Ángulo-Lado



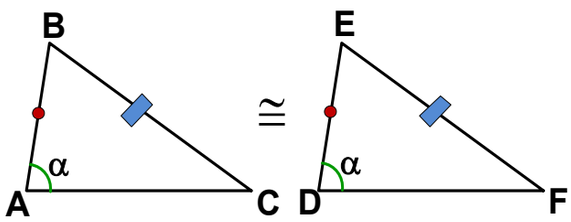

### Ángulo-Lado-Ángulo




### Lado-Lado-Lado

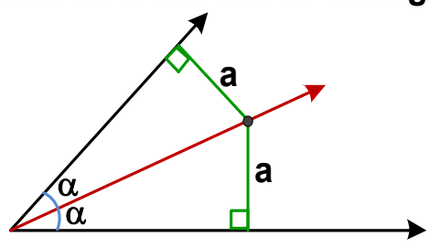



### Lado-Lado-Ángulo mayor



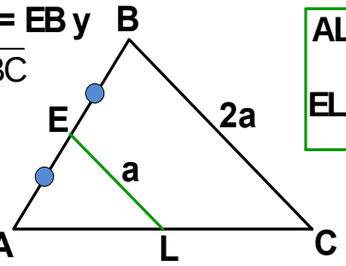

## Aplicaciones de la congruencia

### Teorema de la bisectriz de un ángulo




### Teorema de los puntos medios

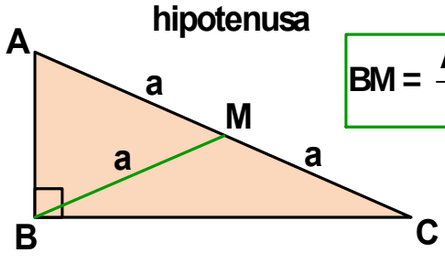
Si AE= EB y $\overline{EL} \parallel \overline{BC}$




$AL = LC$

$EL = \frac{BC}{2}$

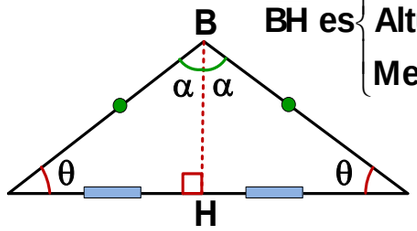
### Teorema de la mediana relativa a la hipotenusa



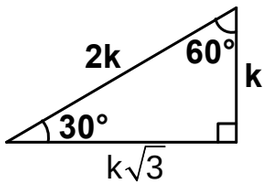
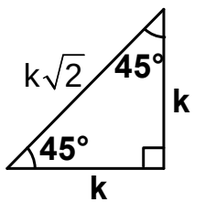
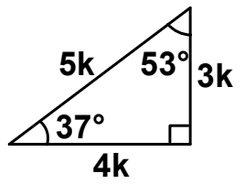
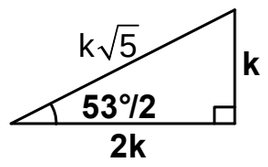
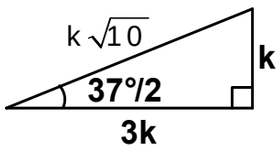
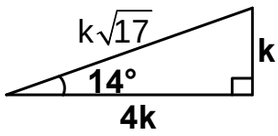
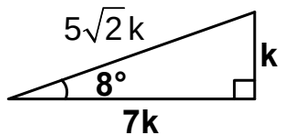
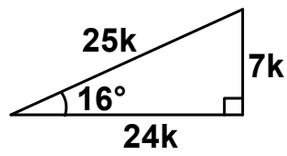
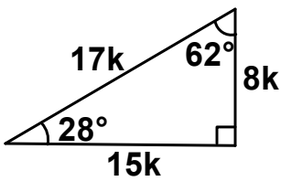
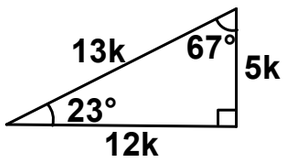
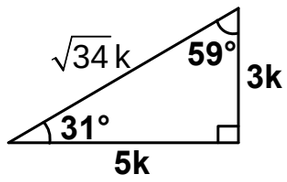
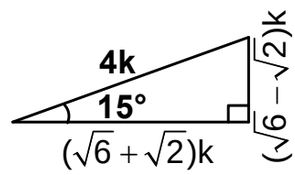
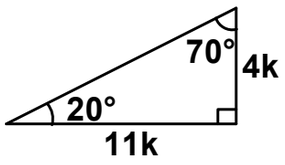
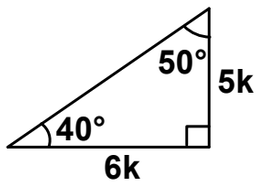
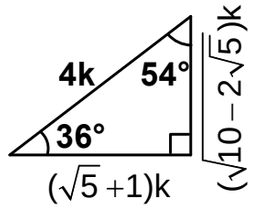
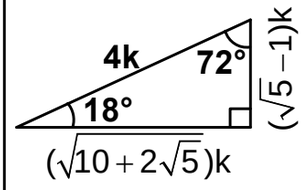

$BM = \frac{AC}{2}$

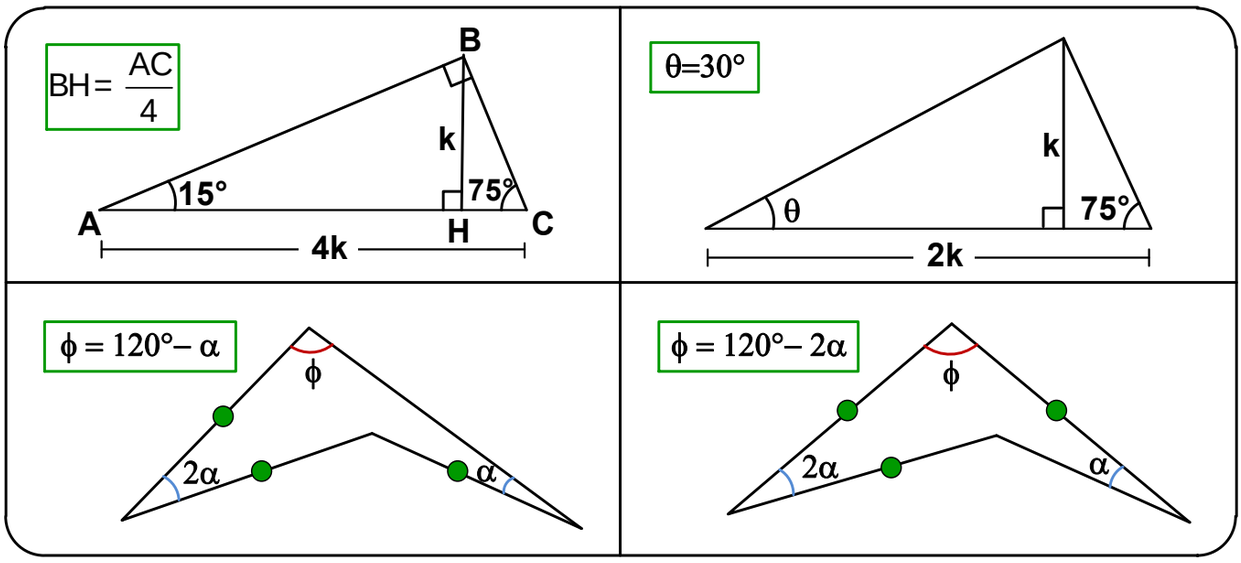
### Triángulo isósceles

$\overline{BH}$ es { Bisectriz, Altura, Mediana }

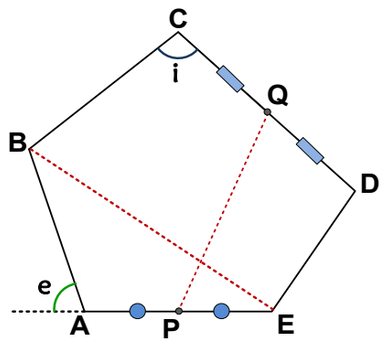


## TRIÁNGULOS RECTÁNGULOS NOTABLES

| De 30° y 60° | De 45° y 45° | De 37° y 53° | De 53°/2 y 127°/2 |
|---|---|---|---|
|   |   |   |   |
| **De 37°/2 y 143°/2** | **De 14° y 76°** | **De 8° y 82°** | **De 16° y 74°** |
|   |   |   |   |
| **De 28° y 62°** | **De 23° y 67°** | **De 31° y 59°** | **De 15° y 75°** |
|   |   |   |   |
| **De 20° y 70°** | **De 40° y 50°** | **De 36° y 54°** | **De 18° y 72°** |
|   |   |   |   |



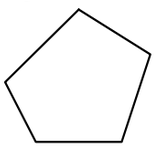
## POLÍGONOS I



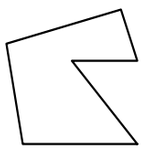

Vértices: A, B, C, D y E
Lados: $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}$ y $\overline{EA}$
Medida de un ángulo interior: *i*
Medida de un ángulo exterior: *e*
Diagonal: $\overline{BE}$
Diagonal media: $\overline{PQ}$

### Clases de polígonos

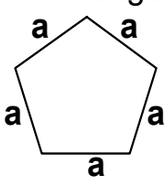
Polígono Convexo

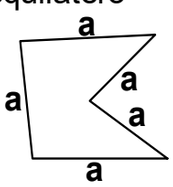


Polígono no convexo

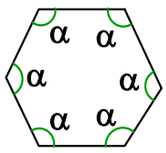


Polígono equilátero

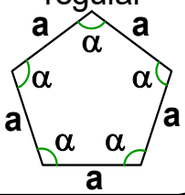



Polígono equiángulo




Polígono regular




### Nombre de algunos polígonos

| N° de lados | Nombre | N° de lados | Nombre |
|---|---|---|---|
| 3 | Triángulo | 13 | Tridecágono |
| 4 | Cuadrilátero | 14 | Tetradecágono |
| 5 | Pentágono | 15 | Pentadecágono |
| 6 | Hexágono | 16 | Hexadecágono |
| 7 | Heptágono | 17 | Heptadecágono |
| 8 | Octógono | 18 | Octadecágono |
| 9 | Nonágono | 19 | Nonadecágono |
| 10 | Decágono | 20 | Icoságono |
| 11 | Undecágono | 30 | Triacontágono |
| 12 | Dodecágono | 40 | Tetracontágono |

| | | |
|---|---|---|
| Diag. trazadas de 1 vértice<br>$N°D_{1vertice} = n-3$ | Diag. medias trazadas de 1 pto medio<br>$N°D_{M\,1pto\,medio} = n-1$ | Diag. trazadas de k vértice consec.<br>$N°D_k = nk - \frac{(k+1)(k+2)}{2}$ |
| N° total de diagonales<br>$N°D = \frac{n(n-3)}{2}$ | N° total de diagonales medias<br>$N°D_M = \frac{n(n-1)}{2}$ | Diag. medias traz. de m ptos medios consec.<br>$N°D_m = nm - \frac{m(m+1)}{2}$ |

# POLÍGONOS II

## Propiedades relativas a los ángulos de un polígono

Suma de medidas de ángulos interiores de un polígono

$$\sum m\measuredangle i = 180°(n-2)$$

Suma de medidas de ángulos exteriores de un polígono convexo

$$\sum m\measuredangle e = 360°$$

Medida de un ángulo interior de un polígono equiángulo

$$m\measuredangle i = \frac{180°(n-2)}{n}$$

Medida de un ángulo exterior de un polígono equiángulo

$$m\measuredangle e = \frac{360°}{n}$$

Medida de un ángulo central de un polígono regular

$$m\measuredangle c = \frac{360°}{n}$$

**Si el polígono es equiángulo o regular**

| N° lados | 5 | 6 | 8 | 10 |
|---|---|---|---|---|
| $m\measuredangle i$ | 108° | 120° | 135° | 144° |
| $m\measuredangle e$ | 72° | 60° | 45° | 36° |

Número de ángulos rectos a que equivale la suma de las medidas de los ángulos interiores de un polígono

$$N°\measuredangle rectos = 2(n-2)$$

Número de triángulos que se forman al trazar las diagonales desde un vértice

$$N°\triangle s = n-2$$

Máximo número de ángulos externos obtusos de un polígono convexo

$$N°max\measuredangle ext.obtusos = 3$$

Máximo número de ángulos internos agudos de un polígono convexo

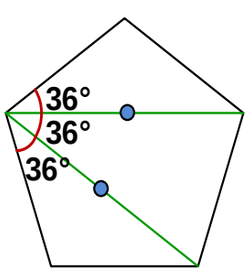
$$N°max\measuredangle int.agudos = 3$$

Mínimo número de ángulos externos agudos de un polígono convexo

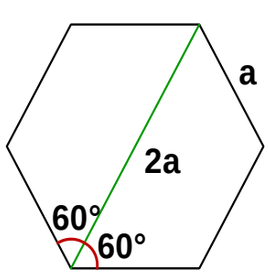
$$N°min\measuredangle ext.agudos = n-3$$

Mínimo número de ángulos internos obtusos de un polígono convexo

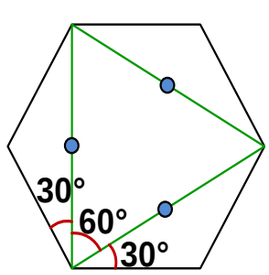
$$N°min\measuredangle int.obtusos = n-3$$



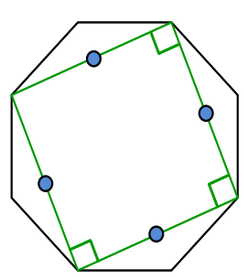
Pentágono regular



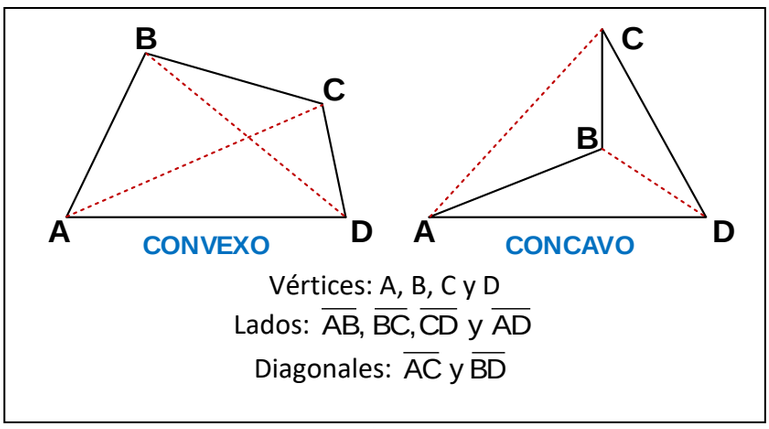
Hexágono regular



Hexágono regular


Octógono regular

## CUADRILÁTEROS I




Vértices: A, B, C y D

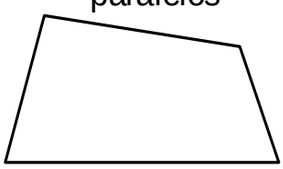
Lados: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ y $\overline{AD}$

Diagonales: $\overline{AC}$ y $\overline{BD}$

## CLASIFICACIÓN DE LOS CUADRILÁTEROS CONVEXOS

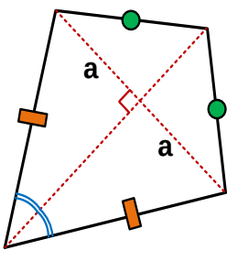
### TRAPEZOIDE

No tiene lados opuestos paralelos

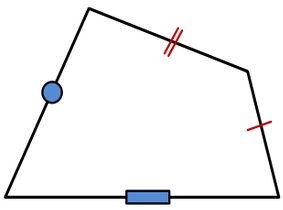


Pueden ser:

Simétrico




Asimétrico

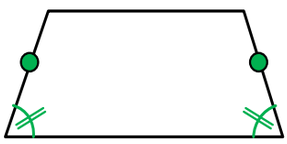


### TRAPECIO

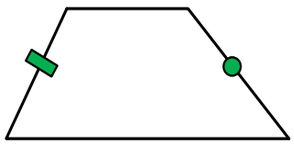
Tiene un par de lados opuestos paralelos

Pueden ser:

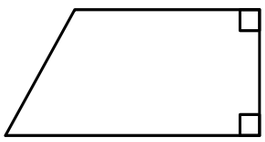
Isósceles



Escaleno

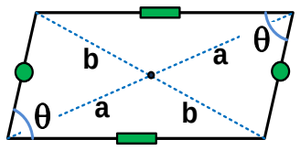


Rectángulo



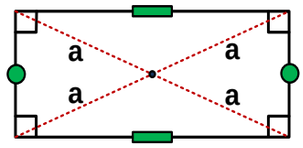
### PARALELOGRAMO

Sus dos pares de lados opuestos son paralelos

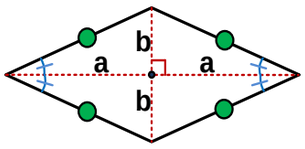



Pueden ser:

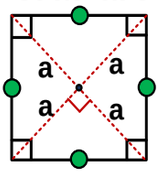
Rectángulo

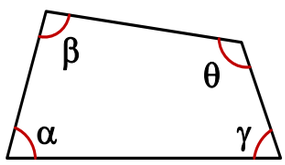



Rombo

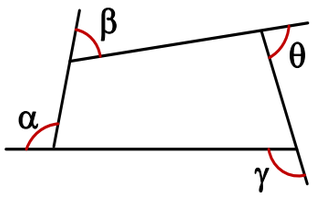



Cuadrado

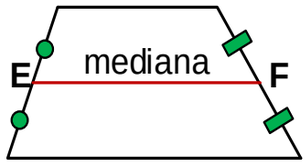



$\alpha+\beta+\theta+\gamma=360°$




$\alpha+\beta+\theta+\gamma=360°$

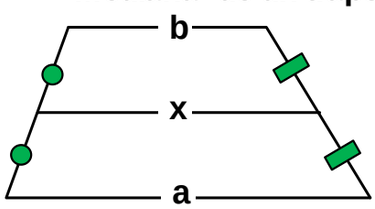



$\overline{EF}$ es paralelo a las bases

## CUADRILÁTEROS II

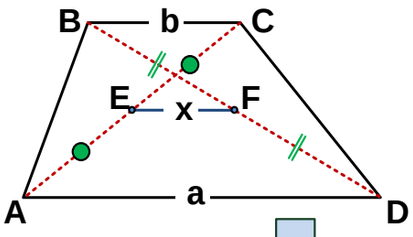
# PROPIEDADES DE LOS CUADRILÁTEROS CONVEXOS

### Mediana de un trapecio



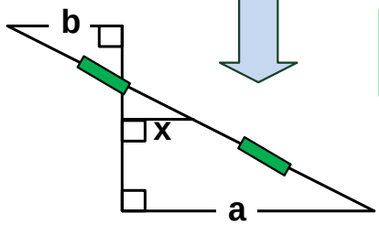

$$x = \frac{a+b}{2}$$

### Segmento que une los puntos medios de las diagonales de un trapecio



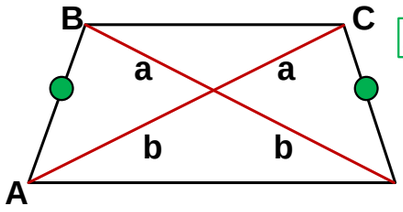

$$x = \frac{a-b}{2}$$

$\overline{EF} \parallel \overline{AD}$



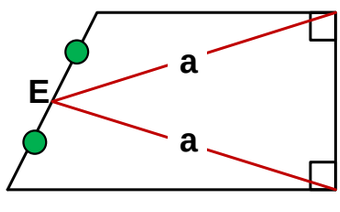

$$x = \frac{a-b}{2}$$

### Las diagonales de un trapecio isósceles son congruentes



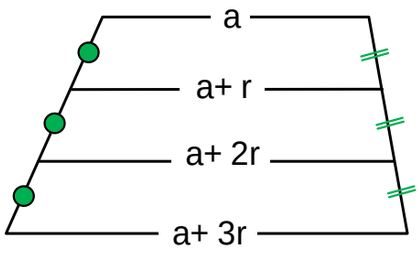

$AC = BD$

### En todo trapecio rectángulo

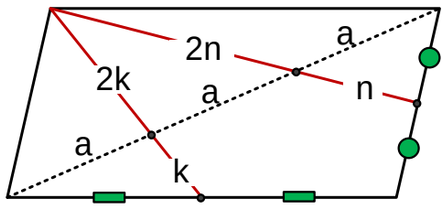



$EL = EN$

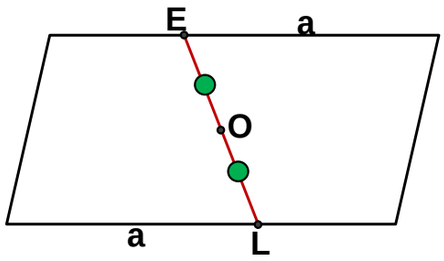
### En todo trapecio



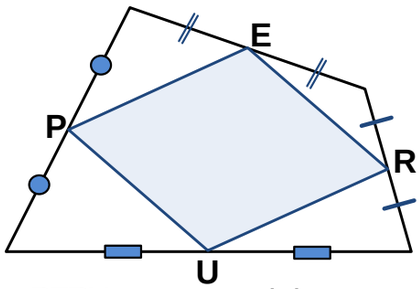

### En todo paralelogramo o romboide




### Si O es centro del paralelogramo o romboide

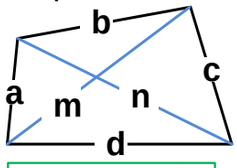



### En todo trapezoide asimétrico



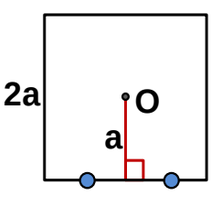

PERU es un paralelogramo

Si $2p = a + b + c + d$

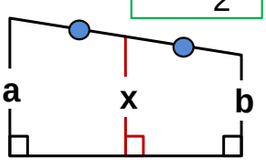



$p < m + n < 2p$

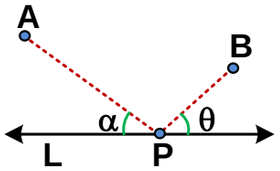
O: centro del cuadrado



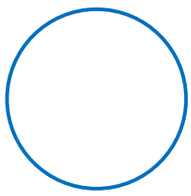

$$x = \frac{a+b}{2}$$



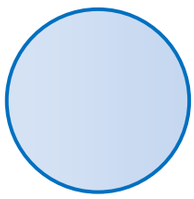

AP+ PB es el menor recorrido $\Leftrightarrow \alpha = \theta$



# CIRCUNFERENCIA



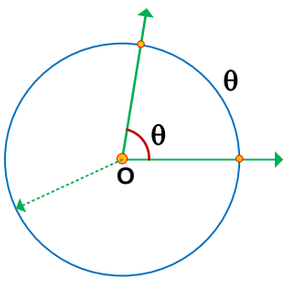
*circunferencia*



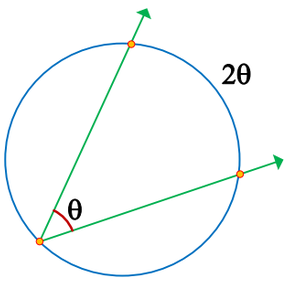
*círculo*

## I. ÁNGULOS ASOCIADOS A LA CIRCUNFERENCIA

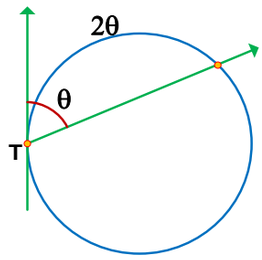
**Ángulo central**




**Ángulo inscrito**

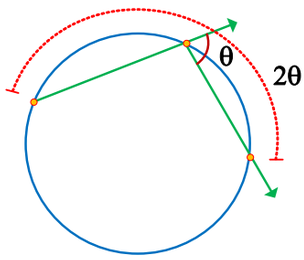



**Ángulo semi-inscrito**

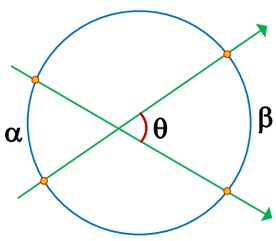



*T: punto de tangencia*

**Ángulo ex-inscrito**

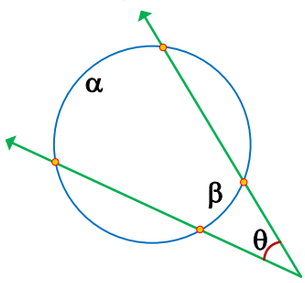



**Ángulo interior**



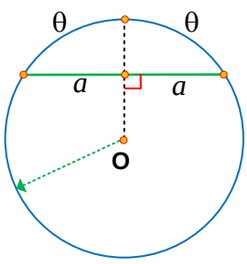

$$\theta = \frac{\alpha+\beta}{2}$$

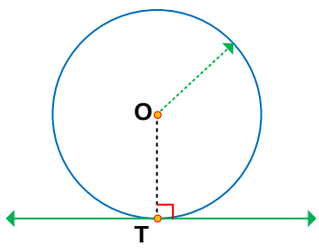
**Ángulo exterior**



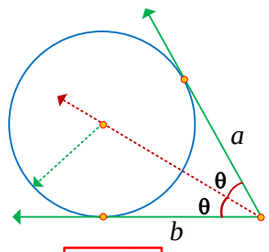

$$\theta = \frac{\alpha-\beta}{2}$$

## II. TEOREMAS ASOCIADOS A LA CIRCUNFERENCIA

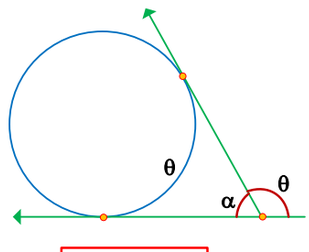


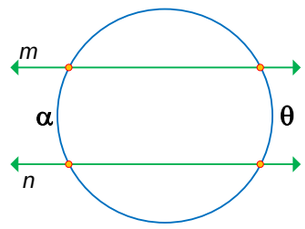

*T: punto de tangencia*



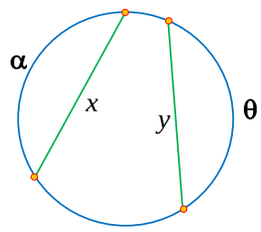

$$a = b$$




$$\alpha + \theta = 180°$$

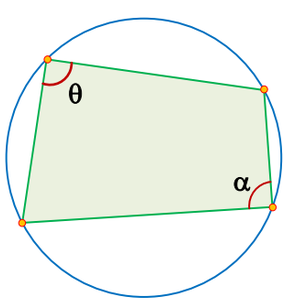



$$m /\!/ n \Leftrightarrow \alpha = \theta$$

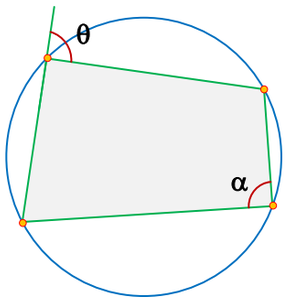



$$x = y \Leftrightarrow \alpha = \theta$$

## III. CUADRILÁTERO INSCRITO

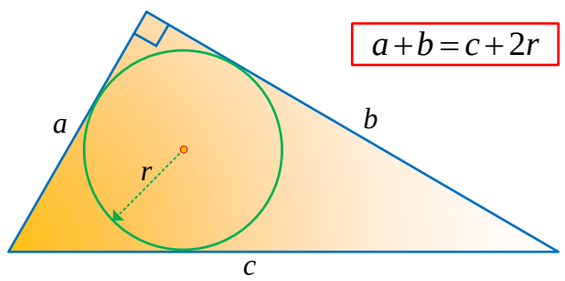



$$\alpha + \theta = 180°$$



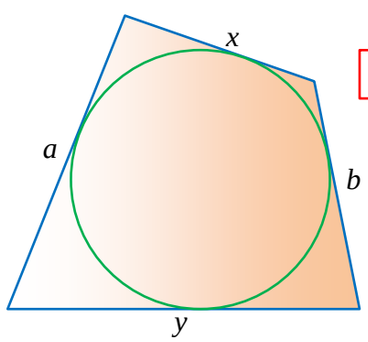

$$\alpha = \theta$$

**TEOREMA DE PONCELET**




$$a + b = c + 2r$$

**TEOREMA DE PITOT**

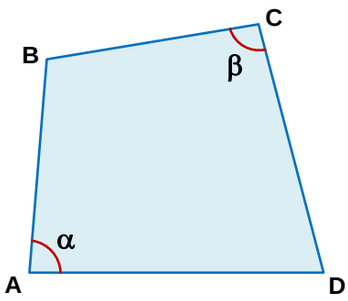



$$a + b = x + y$$

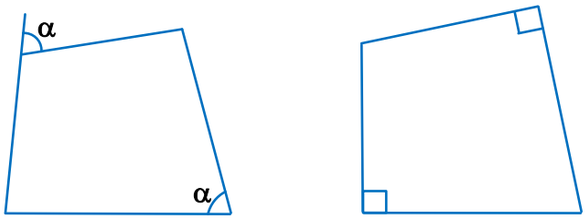
## IV. CUADRILÁTERO INSCRIPTIBLE

Se llama **CUADRILÁTERO INSCRIPTIBLE** al cuadrilátero convexo que puede inscribirse en una circunferencia.

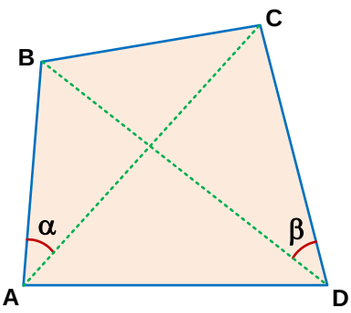
Caso 1. Si $\alpha+\beta=180° \Rightarrow$ ⏢ABCD es inscriptible



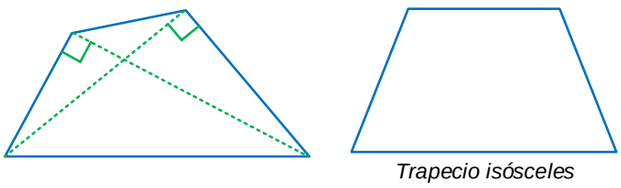

Los siguientes cuadriláteros son inscriptibles.




Caso 2. Si $\alpha=\beta \Rightarrow$ ⏢ABCD es inscriptible



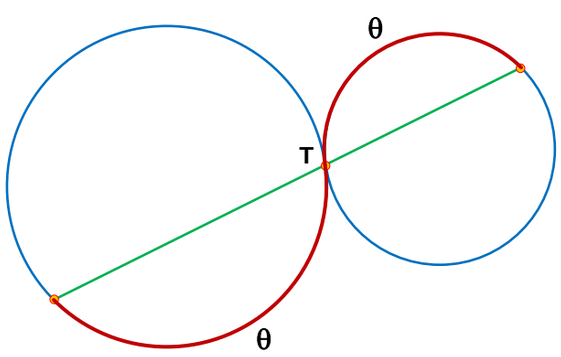

Los siguientes cuadriláteros son inscriptibles.

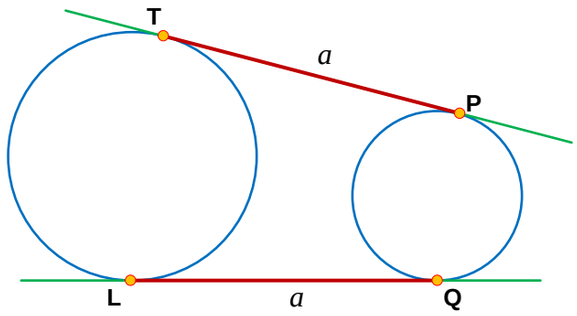


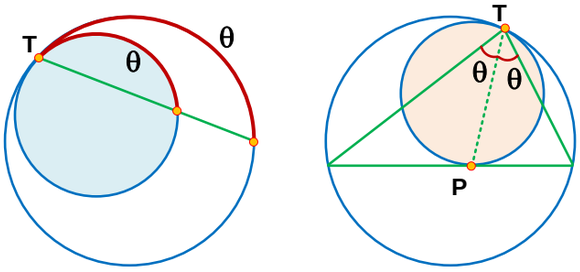
*Trapecio isósceles*

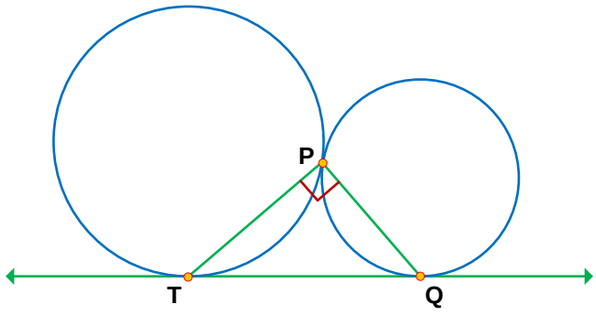
## V. PROPIEDADES ADICIONALES

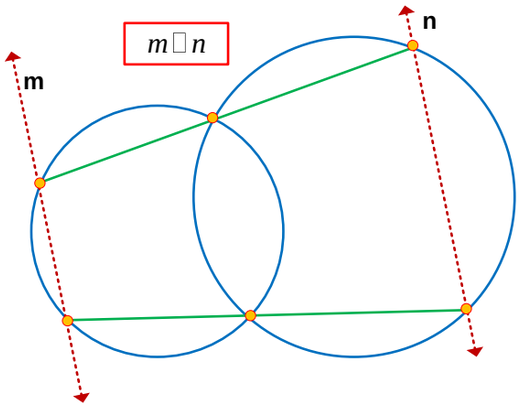
*(T, P, Q y L son puntos de tangencia)*



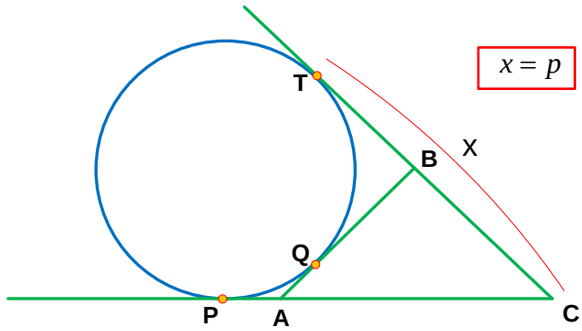



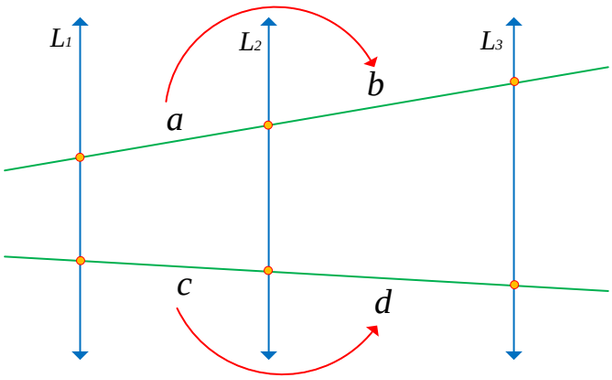

*p es semiperímetro del* △ABC



## PROPORCIONALIDAD Y SEMEJANZA

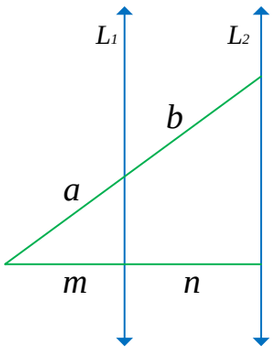
### I. PROPORCIONALIDAD DE SEGMENTOS

**TEOREMA DE THALES** ( $L_1 \parallel L_2 \parallel L_3$ )

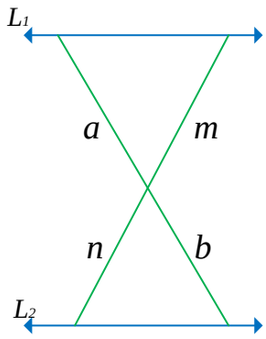



$$\frac{a}{b}=\frac{c}{d} \quad \Leftrightarrow \quad (a)(d)=(b)(c)$$

**COROLARIOS DE THALES** ( $L_1 \parallel L_2$ )



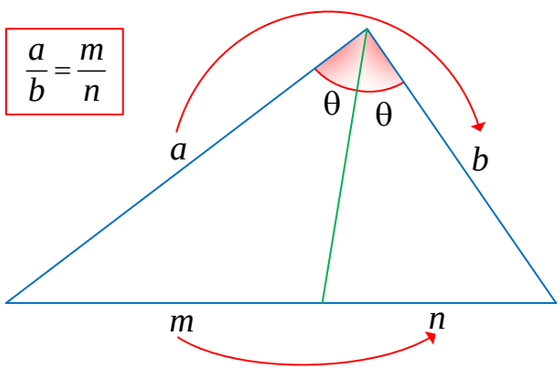

$$\frac{a}{b}=\frac{m}{n}$$




$$\frac{a}{b}=\frac{m}{n}$$

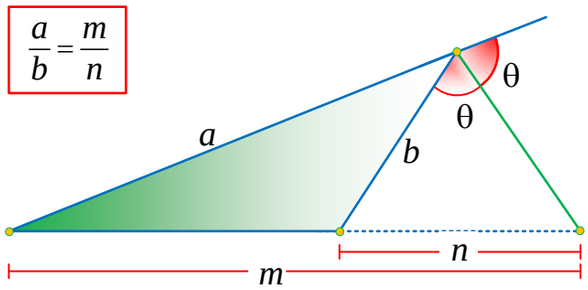
**TEOREMA DE LA BISECTRIZ INTERIOR**

$$\frac{a}{b}=\frac{m}{n}$$



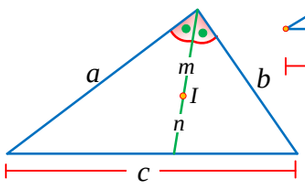

**TEOREMA DE LA BISECTRIZ EXTERIOR**

$$\frac{a}{b}=\frac{m}{n}$$

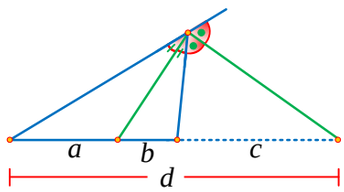



**TEOREMA DEL INCENTRO Y CUATERNA ARMÓNICA**

*Teorema del incentro*
*(I: incentro)*



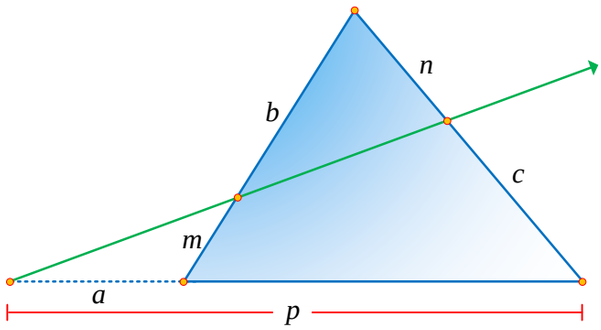

$$\frac{m}{n}=\frac{a+b}{c}$$




*Cuaterna armónica*

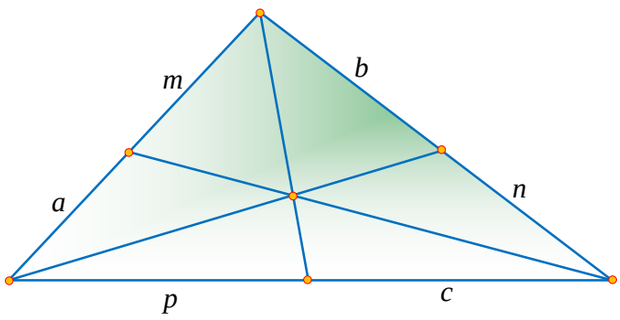
$$\frac{a}{b}=\frac{d}{c}$$

**TEOREMA DE MENELAO**



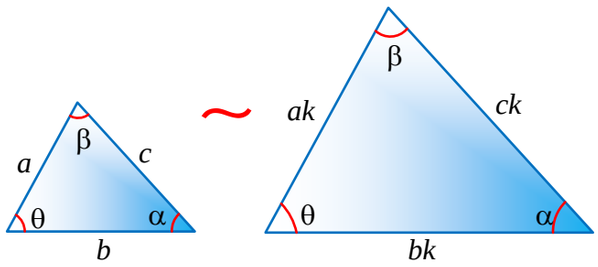

$$(a)(b)(c)=(m)(n)(p)$$

**TEOREMA DE CEVA**

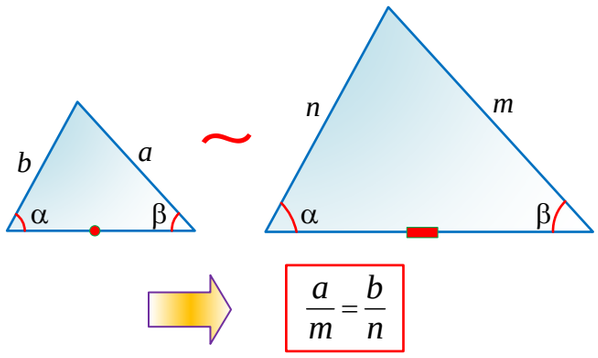



$$(a)(b)(c)=(m)(n)(p)$$

## II. SEMEJANZA DE TRIÁNGULOS

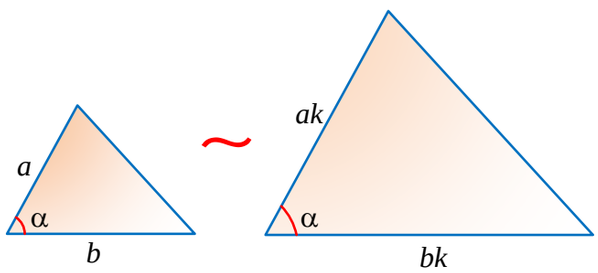



### CASO ÁNGULO – LADO – ÁNGULO (ALA)

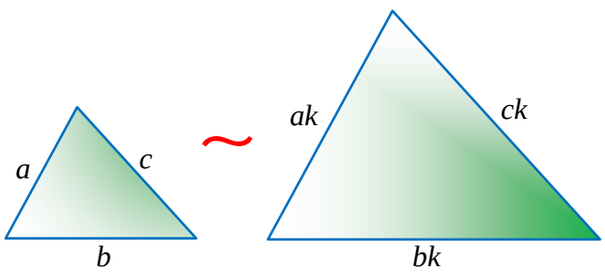



$$\frac{a}{m}=\frac{b}{n}$$

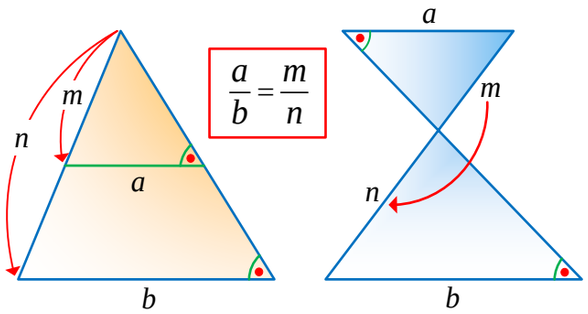
### CASO LADO – ÁNGULO – LADO (LAL)




### CASO LADO – LADO – LADO (LLL)

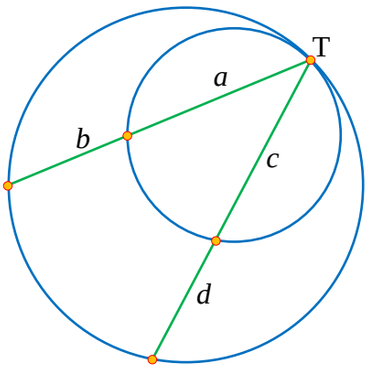



### TEOREMAS



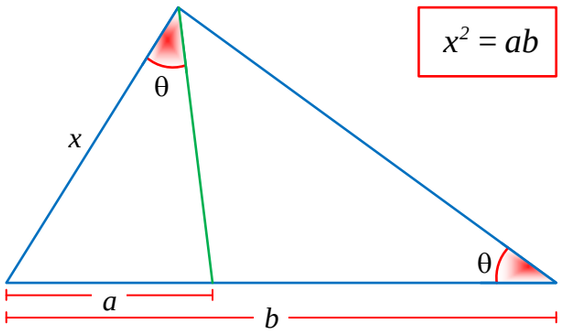

$$\frac{a}{b}=\frac{m}{n}$$

## PROPIEDADES ADICIONALES

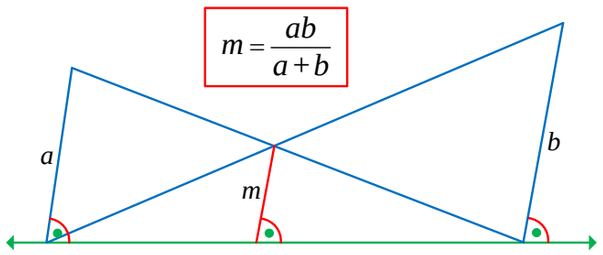



*T: punto de tangencia*

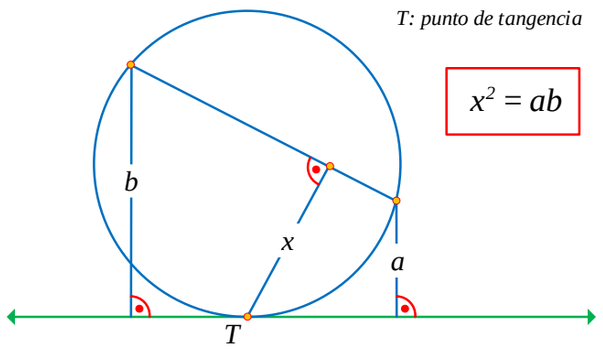
$$\frac{a}{b}=\frac{c}{d}$$




$$x^2=ab$$

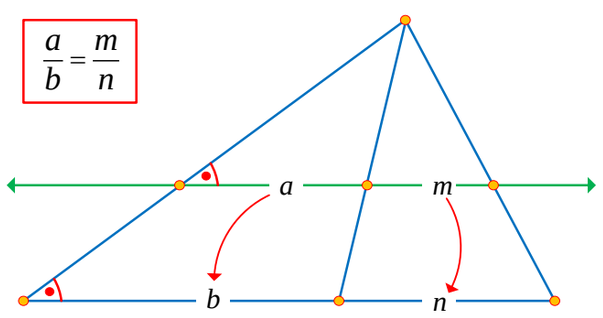



$$m=\frac{ab}{a+b}$$




*T: punto de tangencia*

$$x^2=ab$$

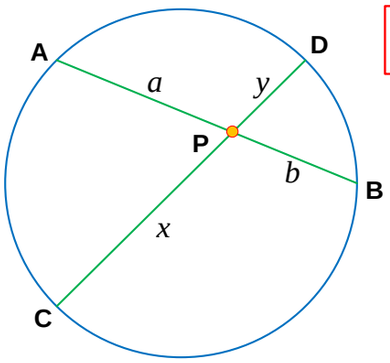



$$\frac{a}{b}=\frac{m}{n}$$

## RELACIONES MÉTRICAS

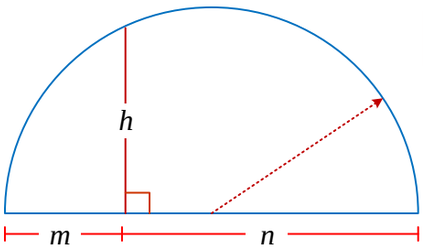
### I. EN LA CIRCUNFERENCIA:

**TEOREMA DE LAS CUERDAS**



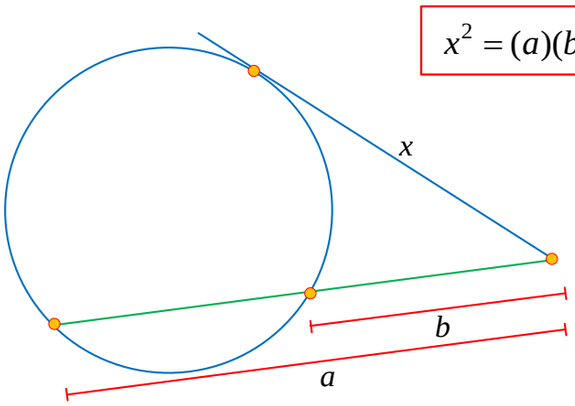

$$(a)(b)=(x)(y)$$

**CALCULO DE LA SEMICUERDA**



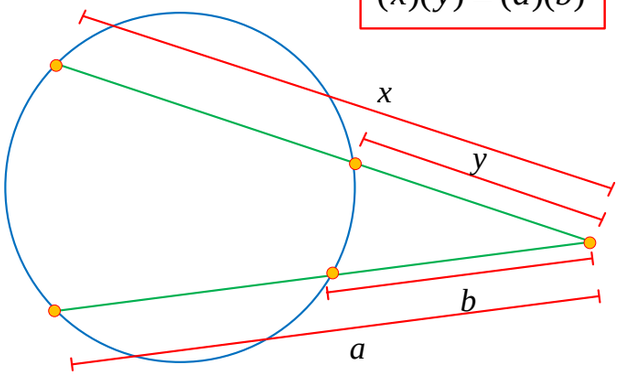

$$h^2=(m)(n)$$

**TEOREMA DE LA TANGENTE**



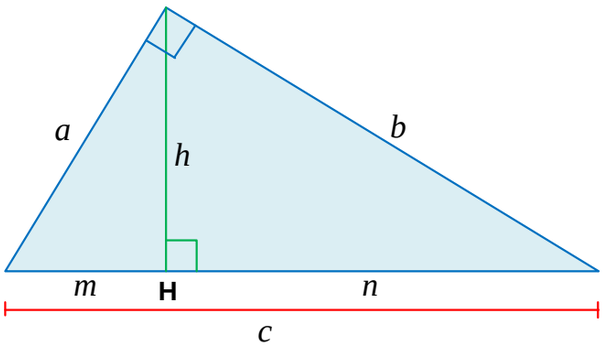

$$x^2=(a)(b)$$

**TEOREMA DE LAS SECANTES**




$$(x)(y)=(a)(b)$$

### II. EN EL TRIÁNGULO:




$$a^2=c.m \qquad b^2=c.n$$

$$a^2+b^2=c^2$$

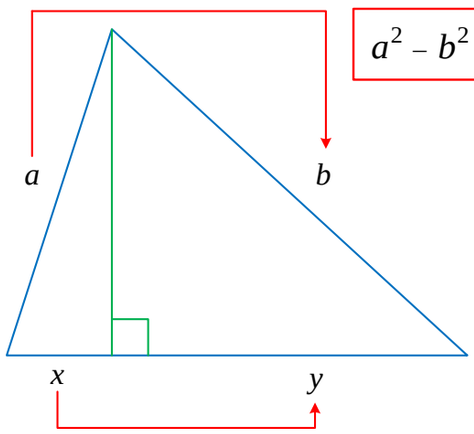
$$h^2=(m)(n)$$

$$\frac{1}{h^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

$$\frac{a^2}{b^2}=\frac{m}{n}$$

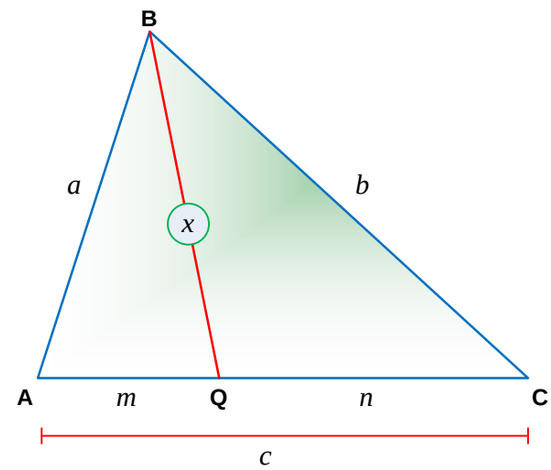
$$(a)(b)=(c)(h)$$

**TEOREMA DE LAS PROYECCIONES**




$$a^2-b^2=x^2-y^2$$

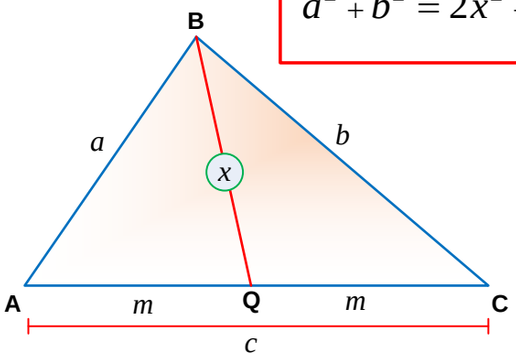
**CÁLCULO DE LA CEVIANA**




$$a^2(n)+b^2(m)=x^2(c)+(m)(n)(c)$$

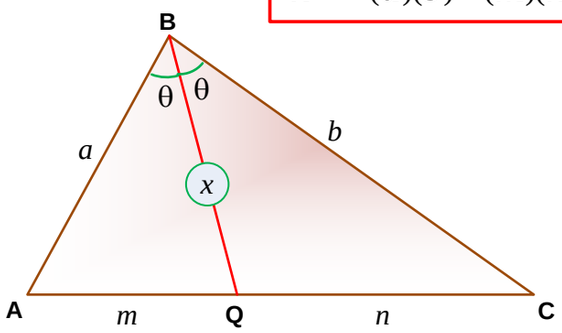
### CÁLCULO DE LA MEDIANA

$$a^2+b^2=2x^2+\frac{c^2}{2}$$



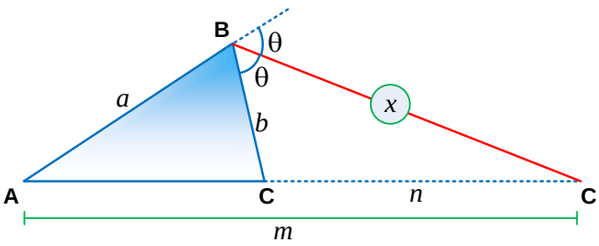

### CÁLCULO DE LA BISECTRIZ INTERIOR

$$x^2=(a)(b)-(m)(n)$$

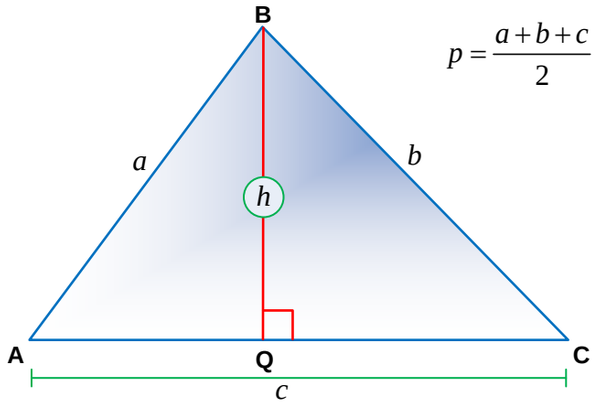



### CÁLCULO DE LA BISECTRIZ EXTERIOR

$$x^2=(m)(n)-(a)(b)$$




### CÁLCULO DE LA ALTURA



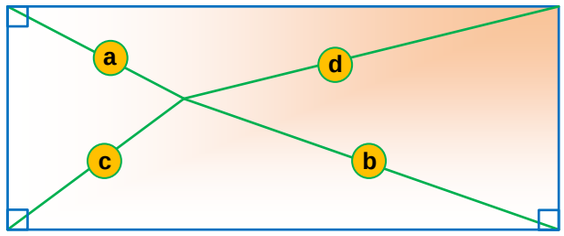

$$p=\frac{a+b+c}{2}$$

$$h=\frac{2}{c}\sqrt{p(p-a)(p-b)(p-c)}$$

## III. EN EL CUADRILATERO:

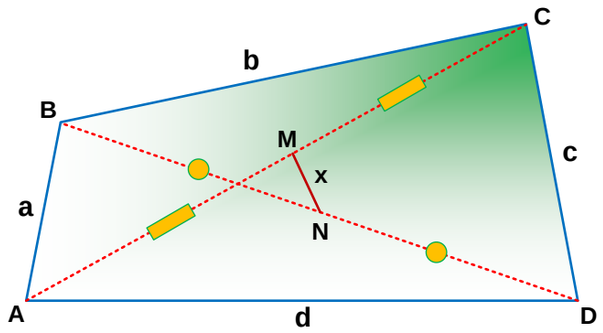
### TEOREMA DE MARLEN

$$a^2+b^2=c^2+d^2$$



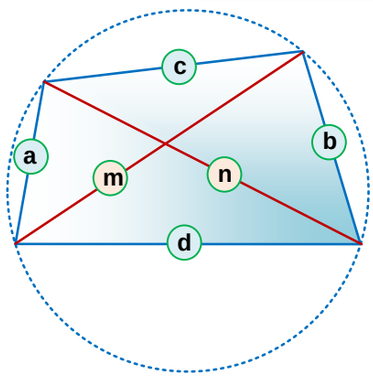

### TEOREMA DE EULER

$$a^2+b^2+c^2+d^2=AC^2+BD^2+4x^2$$




### TEOREMA DE PTOLOMEO

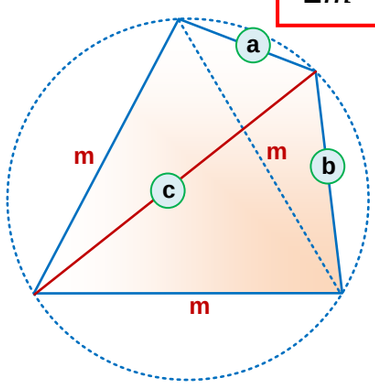
$$(m)(n)=(a)(b)+(c)(d)$$




### TEOREMA DE CHADÚ

$$2m^2=a^2+b^2+c^2$$

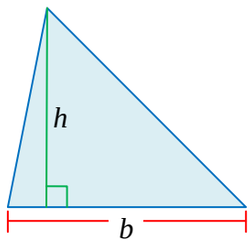
$$c=a+b$$



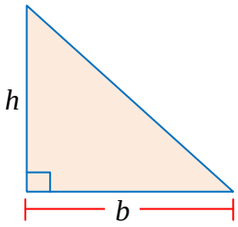
## ÁREAS DE REGIONES PLANAS

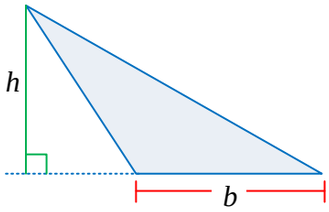
### I. ÁREAS DE REGIONES TRIANGULARES

**FÓRMULA GEOMÉTRICA**

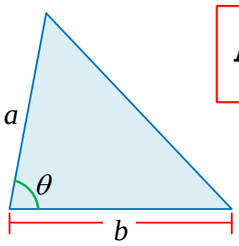



$$\mathbb{A}_{\triangle}=\frac{1}{2}bh$$

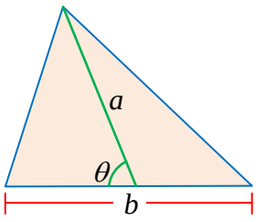


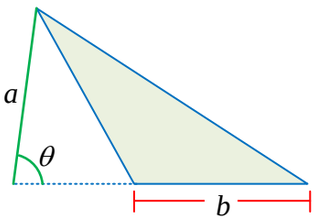




**FÓRMULA TRIGONOMÉTRICA**

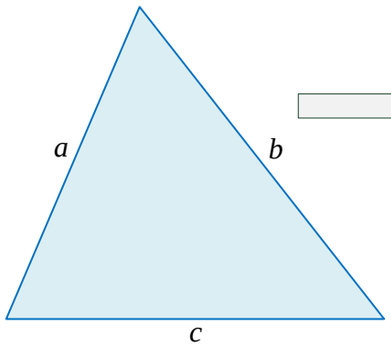


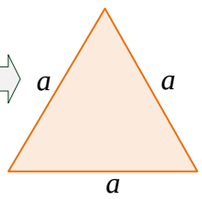

$$\mathbb{A}_{\triangle}=\frac{1}{2}absen\theta$$




**FÓRMULA DE HERÓN**



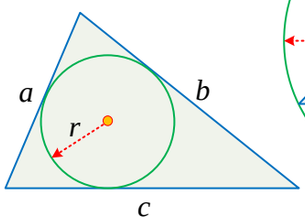

$$\mathbb{A}_{\triangle}=\frac{a^2\sqrt{3}}{4}$$

$$\mathbb{A}_{\triangle}=\sqrt{p(p-a)(p-b)(p-c)}$$

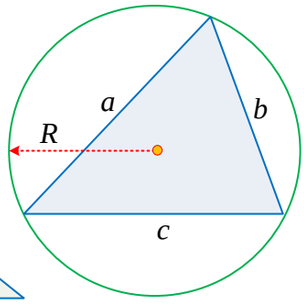
$$p=\frac{a+b+c}{2}\ \text{(semiperimetro)}$$

**FÓRMULA EN FUNCIÓN DEL INRADIO Y CIRCUNRADIO**

$$p=\frac{a+b+c}{2}$$

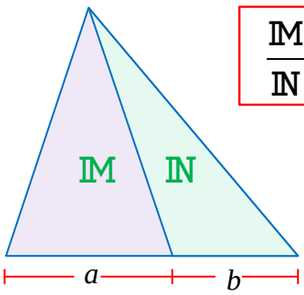



$$\mathbb{A}_{\triangle}=p.r$$

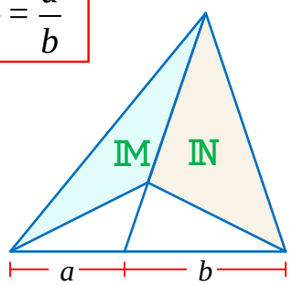



$$\mathbb{A}_{\triangle}=\frac{abc}{4R}$$

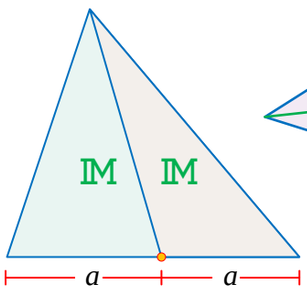
### II. RELACIÓN DE ÁREAS DE REGIONES TRIANGULARES

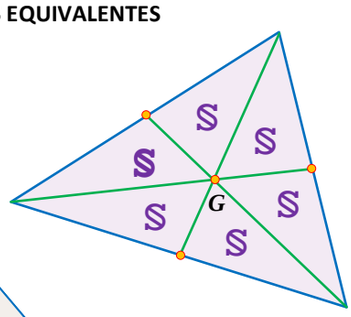



$$\frac{\mathbb{M}}{\mathbb{N}}=\frac{a}{b}$$



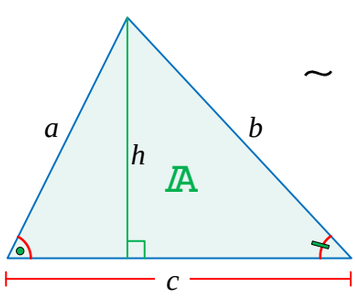

**REGIONES TRIANGULARES EQUIVALENTES**



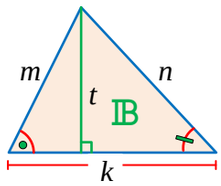




*G: Baricentro*

**REGIONES TRIANGULARES SEMEJANTES**



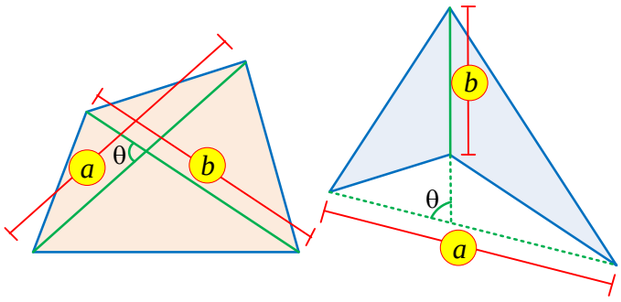

$\sim$




$$\frac{\mathbb{A}}{\mathbb{B}}=\frac{a^2}{m^2}=\frac{b^2}{n^2}=\frac{c^2}{k^2}=\frac{h^2}{t^2}=\ldots$$

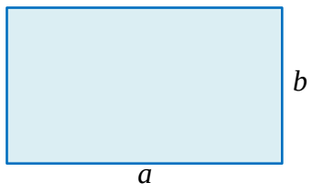
## III. ÁREAS DE REGIONES CUADRANGULARES

$$\mathbb{A}_{\square} = \frac{ab}{2}\,sen\theta$$



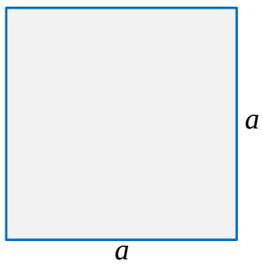

### ÁREA DE UNA REGIÓN RECTANGULAR Y CUADRADA

**Rectangular**




$$\mathbb{A}_{\square} = ab$$

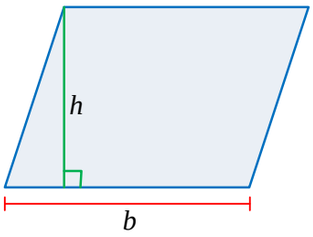
**Cuadrada**




$$\mathbb{A}_{\square} = a^2$$

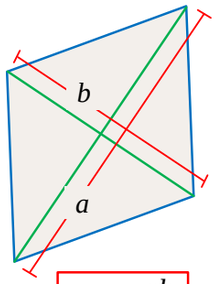
### ÁREA DE UNA REGIÓN ROMBOIDAL Y ROMBAL

**Romboidal**



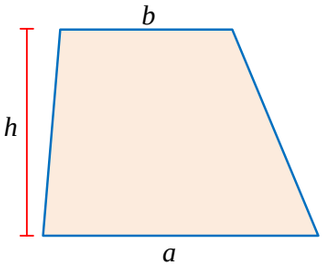

$$\mathbb{A}_{\square} = bh$$

**Rombal**

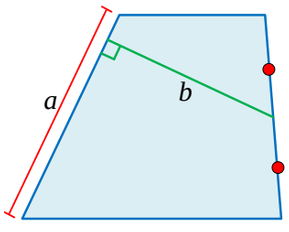



$$\mathbb{A}_{\square} = \frac{ab}{2}$$

### ÁREA DE UNA REGIÓN TRAPECIAL




$$\mathbb{A}_{\square} = \left(\frac{a+b}{2}\right)h$$

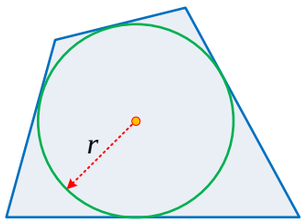



$$\mathbb{A}_{\square} = ab$$

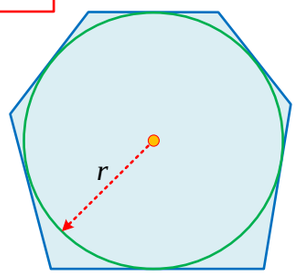
### ÁREA EN FUNCIÓN DEL INRADIO

$$\mathbb{A}_{\square} = p.r$$

**Cuadrilátero circunscrito**



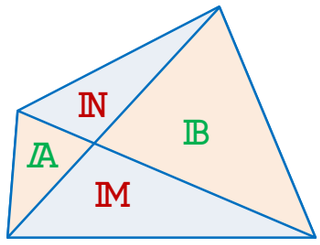

**Polígono circunscrito**




***p*: semiperimetro**

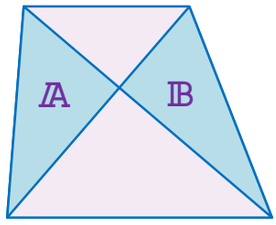
### RELACIÓN DE ÁREAS

**Trapezoide**



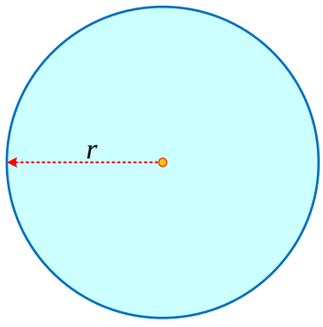

$$\mathbb{A}_{\circ}\mathbb{B} = \mathbb{M}_{\circ}\mathbb{N}$$

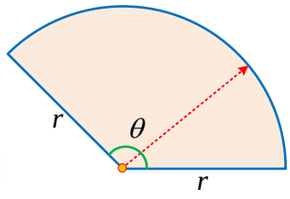
**Trapecio**




$$\mathbb{A} = \mathbb{B}$$

## IV. ÁREAS DE REGIONES CIRCULARES



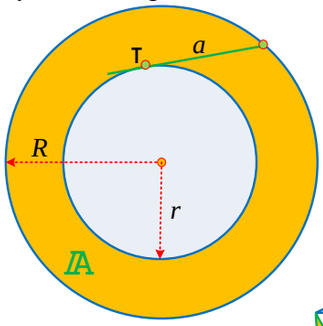

$$\mathbb{A} = \pi r^2 \frac{\theta}{360°}$$

$$Lc = 2\pi r$$

$$\mathbb{A}_{\bigcirc} = \pi r^2$$

### CORONA CIRCICULAR Y LÚNULAS DE HIPOCRATES

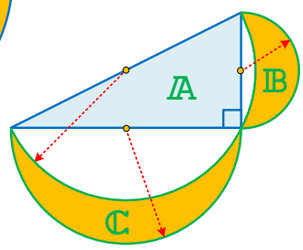
**T: punto de tangencia**




$$\mathbb{A} = \pi(R^2 - r^2)$$

$$\mathbb{A} = \pi a^2$$

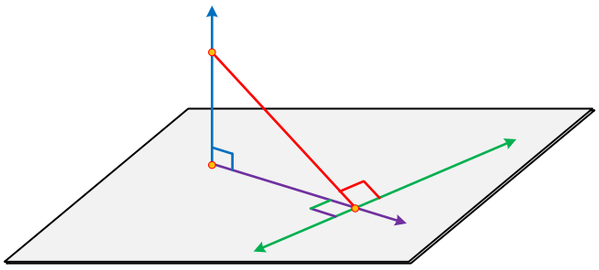
$$\mathbb{A} = \mathbb{B} + \mathbb{C}$$



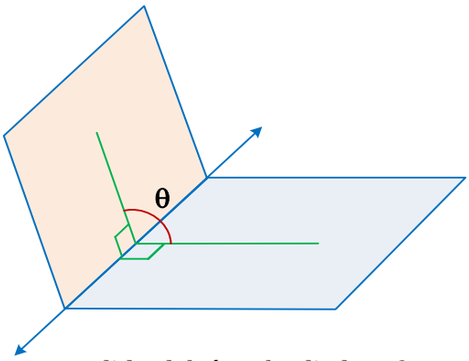
## ELEMENTOS DE GEOMETRÍA DEL ESPACIO

### RECTAS Y PLANOS

**TEOREMA DE LAS TRES PERPENDICULARES**



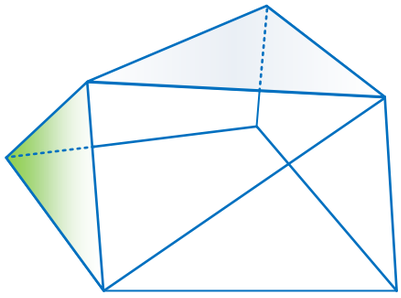
**ÁNGULO DIEDRO**




*medida del ángulo diedro* : $\theta$

### SÓLIDOS POLIEDRICOS

**TEOREMA DE EULER**



$$C + V = A + 2$$

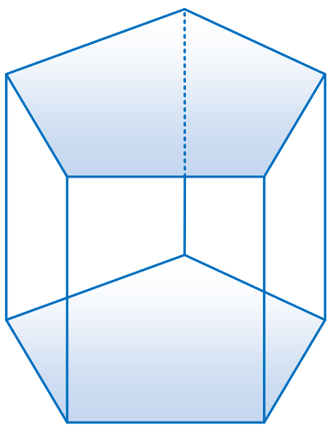
*Donde* :

$C$ : *n° caras*

$V$ : *n° vértices*

$A$ : *n° aristas*

**PRISMA RECTO**

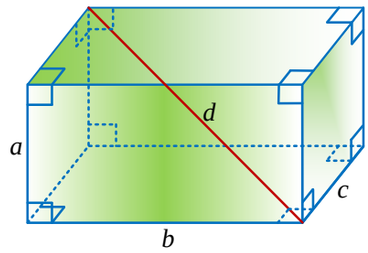



$$A_L = (2p_{base})h$$

$$A_T = A_L + 2A_{base}$$

$$V_{Prisma} = (A_{base})h$$

**PARALELEPÍPEDO RECTANGULAR**
**(Rectoedro u Ortoedro)**

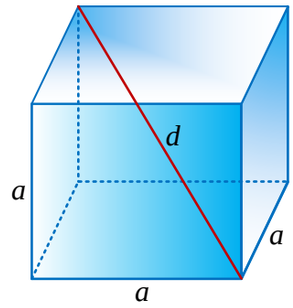



$$A_T = 2(ab + bc + ac)$$

$$V_{Rectoedro} = abc$$

$$d^2 = a^2 + b^2 + c^2$$

**HEXAEDRO REGULAR O CUBO**
**(Caras= 6 regiones cuadradas iguales)**



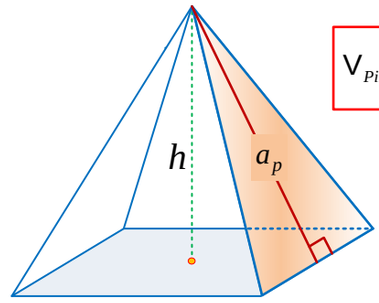

$$A_T = 6a^2$$

$$V_{Cubo} = a^3$$

$$d^2 = 3a^2$$

*Prisma regular* : *Base* = *Región poligonal regular*

**PIRÁMIDE**



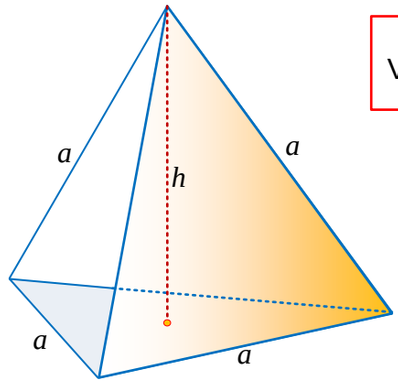

$$V_{Pirámide} = \frac{1}{3}(A_{Base})h$$

$$A_L = (p_{base})a_p$$

$$A_T = A_L + A_{Base}$$

*Pirámide regular* : *Base* = *Región poligonal regular*

**TETRAEDRO REGULAR**
**(Caras= 4 regiones triangulares equiláteras)**




$$V_{Tetraedro} = \frac{a^3\sqrt{2}}{12}$$

$$A_T = a^2\sqrt{3}$$

$$h = \frac{a\sqrt{6}}{3}$$

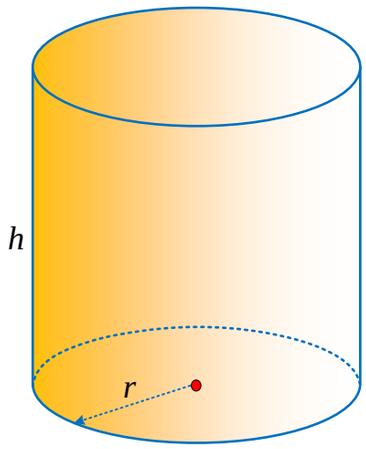
**OCTAEDRO REGULAR**

**(Caras= 8 regiones triangulares equiláteras)**

$$V_{Octaedro} = \frac{a^3\sqrt{2}}{3} \qquad A_T = 2a^2\sqrt{3}$$

## CUERPOS REDONDOS

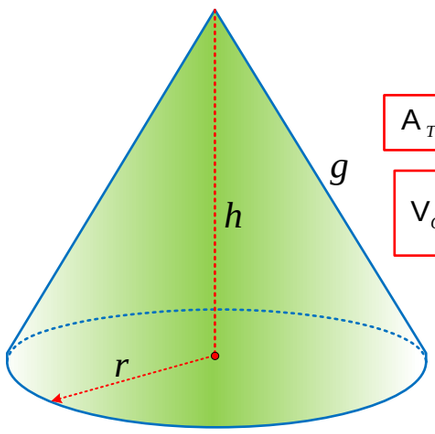
**CILINDRO RECTO O DE REVOLUCIÓN**




$$A_L = 2\pi rh \qquad A_T = 2\pi rh + 2\pi r^2 \qquad V_{Cilindro} = \pi r^2 h$$

*Cilindro equilátero* : $h = 2r$

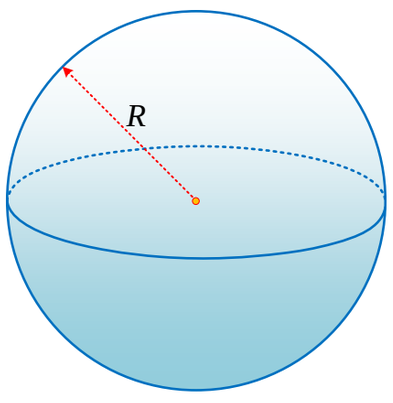
**CONO RECTO O DE REVOLUCIÓN**




$$A_L = \pi rg \qquad A_T = \pi rg + \pi r^2 \qquad V_{Cono} = \frac{1}{3}\pi r^2 h$$

*Cono equilátero* : $g = 2r$

**ESFERA**



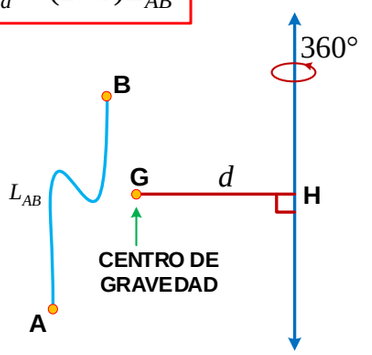

$$A_{SE} = 4\pi R^2 \qquad V_{Esfera} = \frac{4}{3}\pi R^3$$

## TEOREMA DE PAPPUS-GULDIN

**ÁREA DE LA SUPERFICIE GENERADA**

El área de la superficie generada por una línea cuando gira 360° alrededor de una recta, es el producto de la longitud de la circunferencia que describe su centro de gravedad y la longitud de la línea.

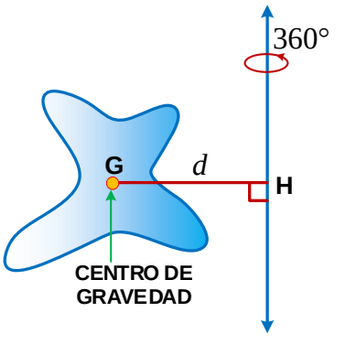
$$A_{Generada} = (2\pi d)L_{AB}$$



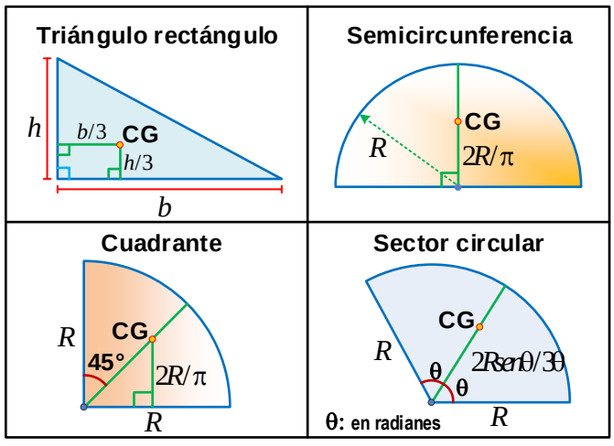

**VOLUMEN DEL SÓLIDO GENERADO**

El volumen del sólido generado por una región plana cuando gira 360° alrededor de una recta, es el producto de la longitud de la circunferencia que describe su centro de gravedad y el área de dicha región.

$$V_{Generado} = (2\pi d)A_{región}$$

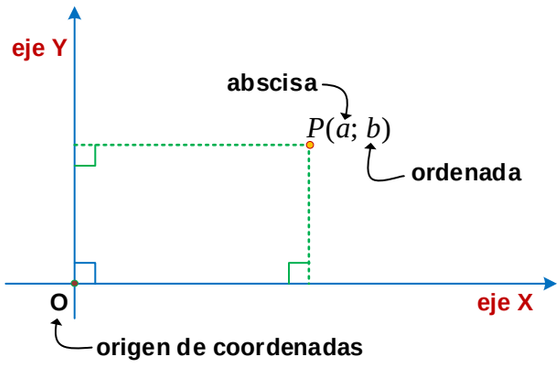



El centro de gravedad (CG) o centroide de un segmento de recta es su punto medio, de un triángulo es su baricentro, de un paralelogramo es el punto de intersección de sus diagonales y de una circunferencia es su centro.



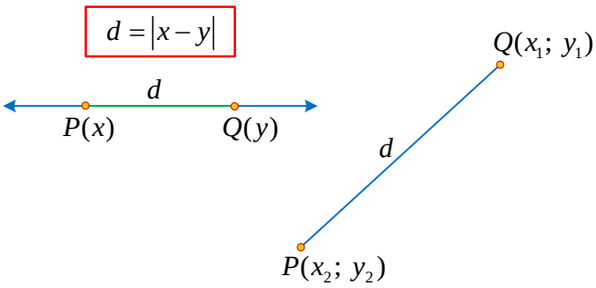
# GEOMETRÍA ANALÍTICA ESCALAR

## I. PLANO CARTESIANO




### DISTANCIA ENTRE DOS PUNTOS

$$d = |x - y|$$

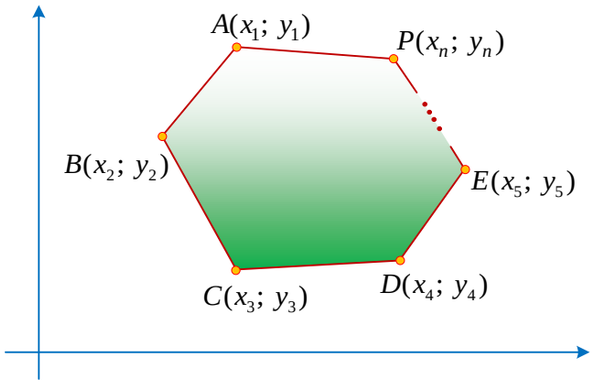






$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

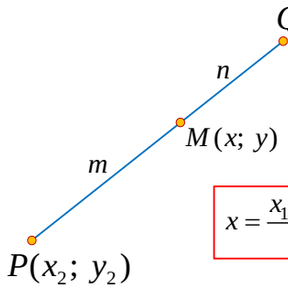
### ÁREA DE UNA REGIÓN POLIGONAL




$$\mathbb{A} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\left[(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (x_2y_1 + x_3y_2 + \ldots + x_1y_n)\right]$$

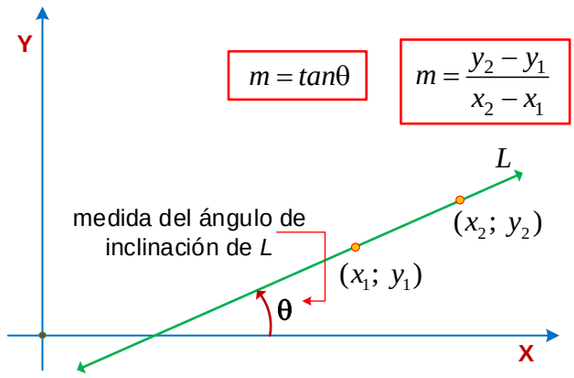
### DIVISIÓN DE UN SEGMENTO EN UNA RAZÓN DADA




$$x = \frac{x_1.m + x_2.n}{m+n} \qquad y = \frac{y_1.m + y_2.n}{m+n}$$

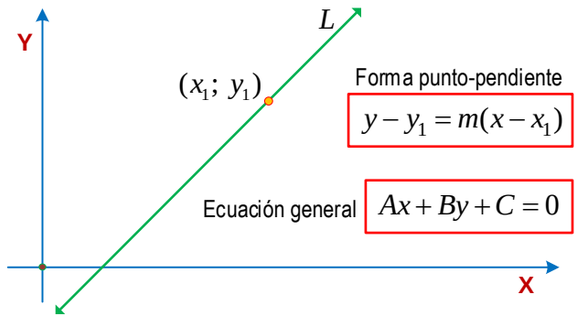
*Si M es punto medio:* $x = \frac{x_1 + x_2}{2}$ $y = \frac{y_1 + y_2}{2}$

### ÁNGULO DE INCLINACIÓN Y PENDIENTE DE UNA RECTA




$$m = tan\theta \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

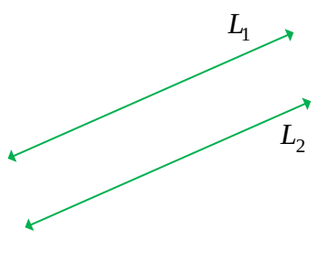
### ECUACIÓN DE LA RECTA

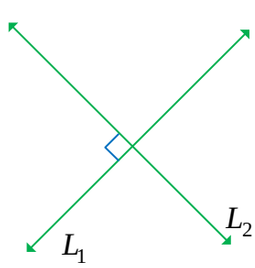



Forma punto-pendiente: $y - y_1 = m(x - x_1)$

Ecuación general: $Ax + By + C = 0$

### RECTAS PARALELAS Y RECTAS PERPENDICULARES

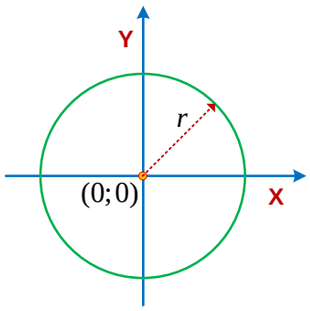



$L_1 \parallel L_2 \Leftrightarrow m_1 = m_2$ $\qquad$ $L_1 \perp L_2 \Leftrightarrow (m_1)(m_2) = -1$

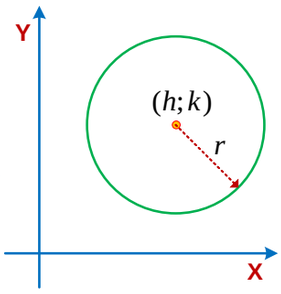
## II. CIRCUNFERENCIA

**Ecuación canónica**




$$x^2+y^2=r^2$$

**Ecuación ordinaria**




$$(x-h)^2+(y-k)^2=r^2$$

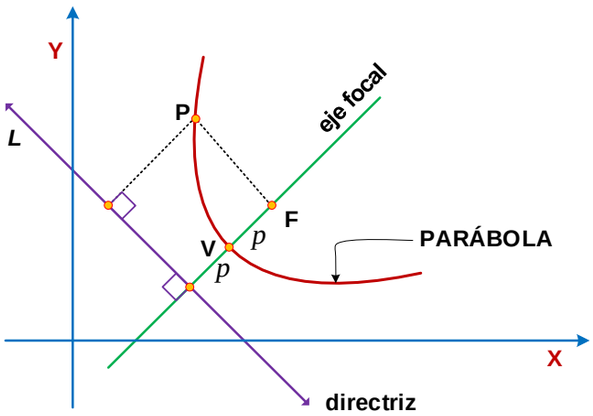
**Ecuación general**

$$x^2+y^2+Ax+By+C=0$$

## III. PARÁBOLA

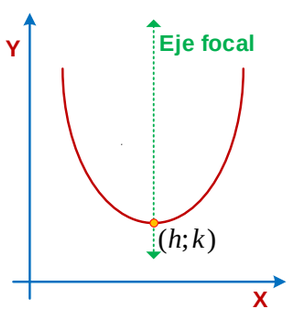
Sea **F** un punto fijo del plano (llamado foco) y **L** una recta fija (llamada directriz).

Una *parábola* es el conjunto de todos los puntos **P** del plano tal que $d(P,F)=d(P,L)$

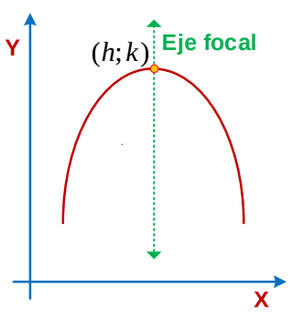



* "p" se llama *parámetro* de la parábola.
* Una *cuerda* es cualquier segmento de recta cuyos extremos son puntos de la parábola. La cuerda perpendicular al eje focal y que pasa por el foco, se llama *lado recto* y su longitud es: $Lr=|4p|$

### ECUACIÓN DE LA PARÁBOLA CON EJE FOCAL || AL EJE Y

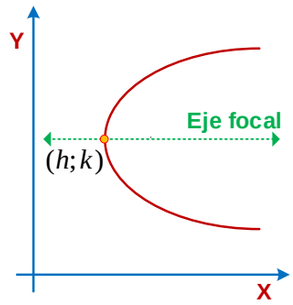



$$(x-h)^2=4p(y-k)$$

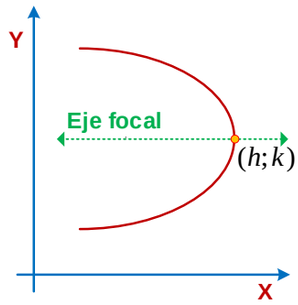



$$(x-h)^2=-4p(y-k)$$

### ECUACIÓN DE LA PARÁBOLA CON EJE FOCAL || AL EJE X




$$(y-k)^2=4p(x-h)$$

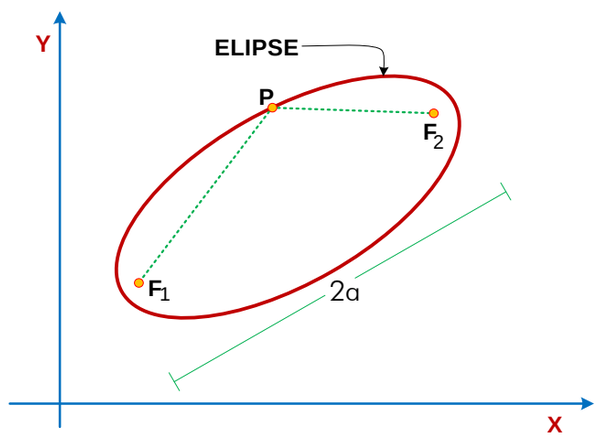



$$(y-k)^2=-4p(x-h)$$

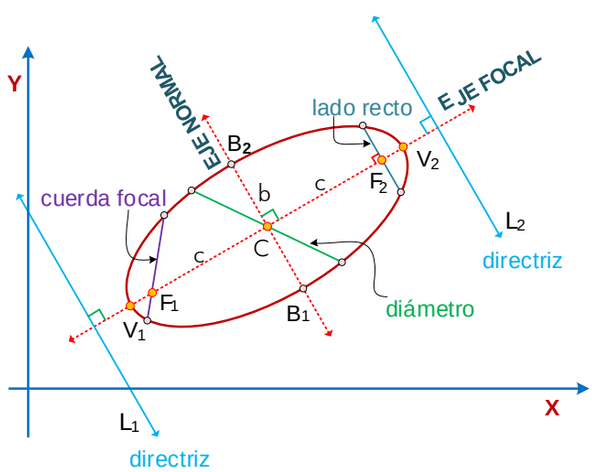
## IV. ELIPSE

Sea $F_1$ y $F_2$ dos puntos fijos del plano (llamados focos) y "a" un número real positivo.

Una *elipse* es el conjunto de todos los puntos **P** del plano tal que, $d(P,F_1)+d(P,F_2)=2a$

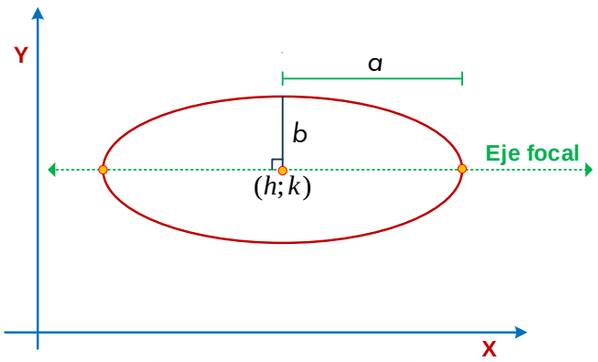



### ELEMENTOS DE LA ELIPSE



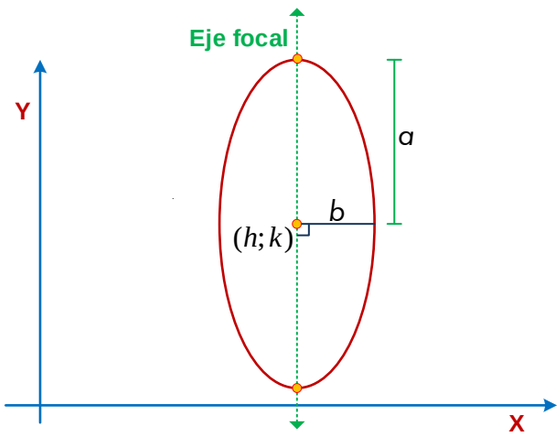

* $\overline{V_1V_2}$ se llama *eje mayor*, tiene longitud 2a.
* $\overline{B_1B_2}$ se llama *eje menor*, tiene longitud 2b.
* Se cumple: $a^2=b^2+c^2$; $a>b$; $a>c$
* Lado recto: $Lr=\dfrac{2b^2}{a}$ ; excentricidad: $e=\dfrac{c}{a}$

**ECUACIÓN DE LA ELIPSE CON EJE FOCAL || AL EJE X**




$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

**ECUACIÓN DE LA ELIPSE CON EJE FOCAL || AL EJE Y**



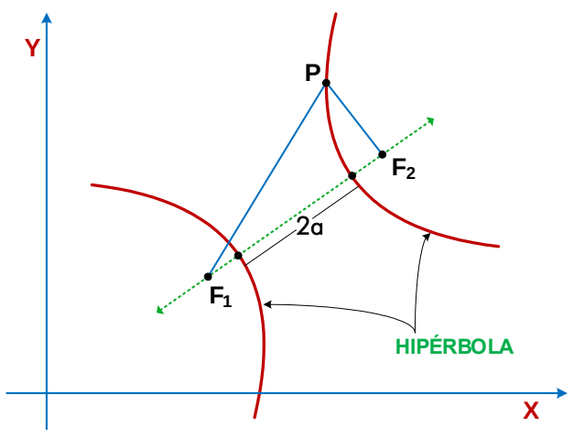

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

## V. HIPÉRBOLA

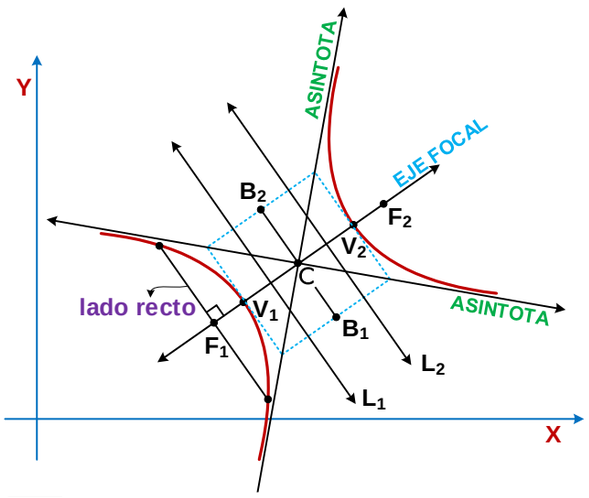
Sea $F_1$ y $F_2$ dos puntos fijos del plano (llamados focos) y **a** un número real positivo.

Una hipérbola es el conjunto de todos los puntos P del plano tal que,

$$\left|d(P,F_1)-d(P,F_2)\right|=2a$$



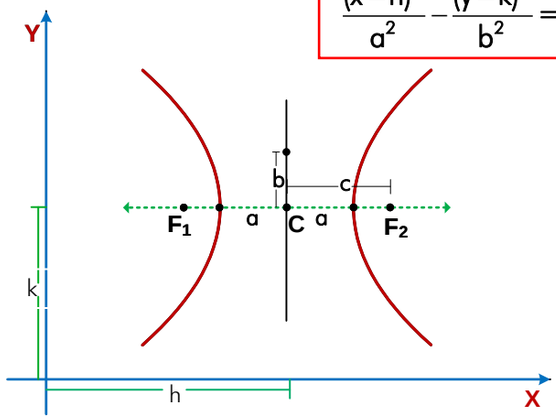

**ELEMENTOS DE LA HIPÉRBOLA**




* $\overline{V_1V_2}$ se llama *eje transverso*, tiene longitud 2a.
* $\overline{B_1B_2}$ se llama *eje conjugado*, tiene longitud 2b.
* Distancia focal: $F_1F_2=2c$ y se cumple: $a^2+b^2=c^2$
* Lado recto: $Lr=\frac{2b^2}{a}$ ; excentricidad: $e=\frac{c}{a}$ ; $e>1$
* Distancia entre las **directrices**: $d(L_1,L_2)=\frac{2a^2}{c}=\frac{2a}{e}$.

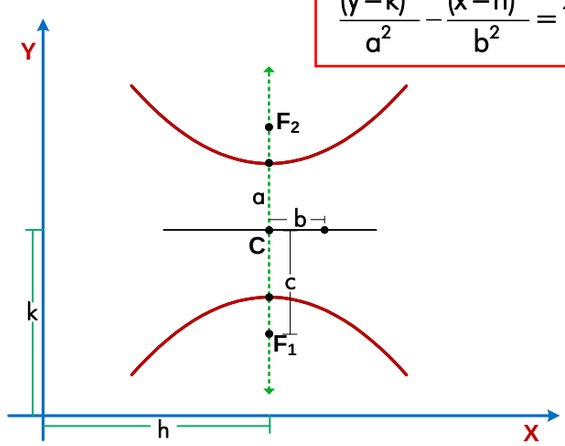
**EC. DE LA HIPÉRBOLA CON EJE FOCAL || AL EJE X**

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$



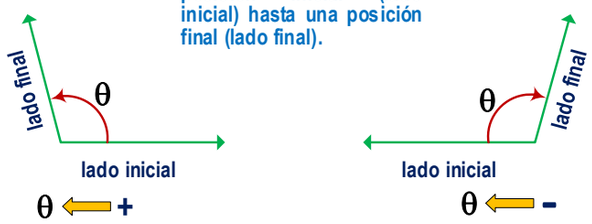

**EC. DE LA HIPÉRBOLA CON EJE FOCAL || AL EJE Y**

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$



## I. ÁNGULO TRIGONOMÉTRICO

Sentido de giro ANTIHORARIO (POSITIVO) ⟵ Se obtiene girando un rayo alrededor de un punto fijo, llamado vértice, desde una posición inicial (lado inicial) hasta una posición final (lado final). ⟶ Sentido de giro HORARIO (NEGATIVO)




## II. SISTEMAS DE MEDIDAS ANGULARES

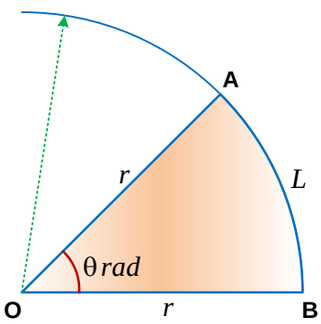
$$\frac{S}{180}=\frac{C}{200}=\frac{R}{\pi} \Rightarrow \begin{cases} S=9k \\ C=10k \\ R=\dfrac{\pi}{20}k \end{cases} \qquad \begin{matrix} 9^\circ = 10^g \\ 180^\circ = \pi\, rad \end{matrix}$$

Donde:
S: número de grados sexagesimales
C: número de grados centesimales
R: número de radianes (1 rad=57° 17′ 45″)
Además,

$1^\circ = 60'\,;\ 1' = 60''\,;\ 1^\circ = 3600''$

$1^g = 100^m\,;\ 1^m = 100^s\,;\ 1^g = 10000^s$

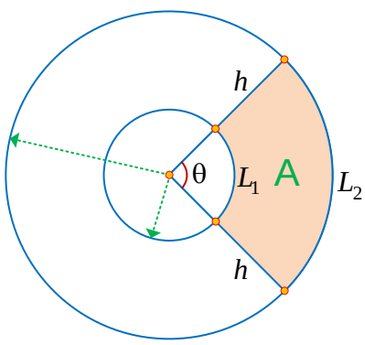
## III. LONGITUD DE ARCO Y ÁREA DE UN SECTOR




$$L = \theta r$$

$$A_{\triangle} = \frac{\theta r^2}{2} = \frac{Lr}{2} = \frac{L^2}{2\theta}$$

### ÁREA DE UN TRAPECIO CIRCULAR

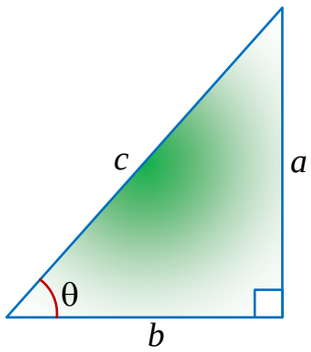



$$A = \left(\frac{L_1 + L_2}{2}\right)h$$

$$\theta = \frac{L_2 - L_1}{h}$$

## IV. RAZONES TRIGONOMÉTRICAS (RT)

Razones trigonométricas de un ángulo agudo en el triángulo rectángulo.




$$\operatorname{sen}\theta = \frac{a}{c} \qquad \csc\theta = \frac{c}{a}$$

$$\cos\theta = \frac{b}{c} \qquad \sec\theta = \frac{c}{b}$$

$$\tan\theta = \frac{a}{b} \qquad \cot\theta = \frac{b}{a}$$

### RT DE ÁNGULOS NOTABLES

| | 30° | 37° | 45° | 53° | 60° |
|---|---|---|---|---|---|
| ***sen*** | $\frac{1}{2}$ | $\frac{3}{5}$ | $\frac{\sqrt{2}}{2}$ | $\frac{4}{5}$ | $\frac{\sqrt{3}}{2}$ |
| ***cos*** | $\frac{\sqrt{3}}{2}$ | $\frac{4}{5}$ | $\frac{\sqrt{2}}{2}$ | $\frac{3}{5}$ | $\frac{1}{2}$ |
| ***tan*** | $\frac{\sqrt{3}}{3}$ | $\frac{3}{4}$ | $1$ | $\frac{4}{3}$ | $\sqrt{3}$ |
| ***cot*** | $\sqrt{3}$ | $\frac{4}{3}$ | $1$ | $\frac{3}{4}$ | $\frac{\sqrt{3}}{3}$ |
| ***sec*** | $\frac{2\sqrt{3}}{3}$ | $\frac{5}{4}$ | $\sqrt{2}$ | $\frac{5}{3}$ | $2$ |
| ***csc*** | $2$ | $\frac{5}{3}$ | $\sqrt{2}$ | $\frac{5}{4}$ | $\frac{2\sqrt{3}}{3}$ |

### RT RECÍPROCAS Y DE ÁNGULOS COMPLEMENTARIOS

α, θ : agudos

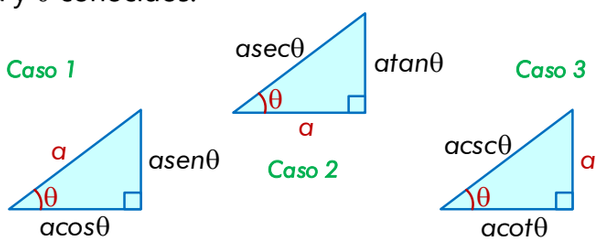
$$\begin{matrix} \operatorname{sen}\theta\csc\theta = 1 \\ \cos\theta\sec\theta = 1 \\ \tan\theta\cot\theta = 1 \end{matrix} \Leftrightarrow \begin{matrix} \csc\theta = \dfrac{1}{\operatorname{sen}\theta} \\ sec\theta = \dfrac{1}{\cos\theta} \\ \cot\theta = \dfrac{1}{\tan\theta} \end{matrix}$$

$$\begin{matrix} \operatorname{sen}\alpha = \cos\theta \\ \tan\alpha = \cot\theta \\ \sec\alpha = \csc\theta \end{matrix} \Leftrightarrow \alpha + \theta = 90^\circ$$

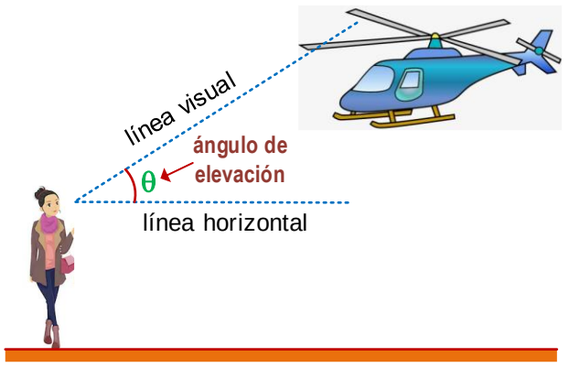
*Tener en cuenta ...* $\tan\theta = \dfrac{\operatorname{sen}\theta}{\cos\theta}\,;\quad \cot\theta = \dfrac{\cos\theta}{\operatorname{sen}\theta}$

### RESOLUCIÓN DE TRIÁNGULOS RECTÁNGULOS

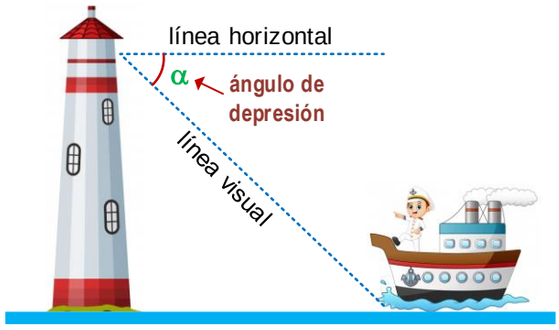
a y θ conocidos.



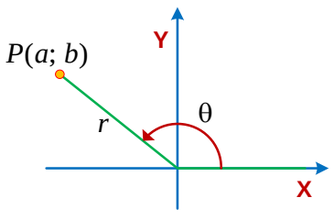

### ÁNGULO DE ELEVACIÓN




### ÁNGULO DE DEPRESIÓN




### RT DE UN ÁNGULO EN POSICIÓN NORMAL



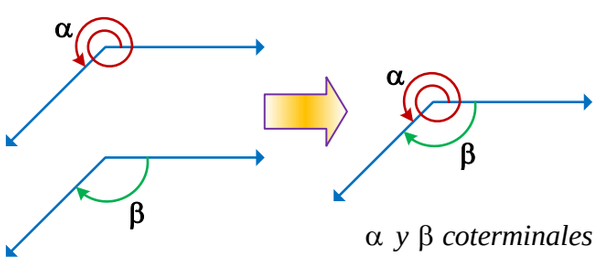

$r^2 = a^2 + b^2$

$$\operatorname{sen}\theta = \frac{b}{r} \qquad \csc\theta = \frac{r}{b}$$

$$\cos\theta = \frac{a}{r} \qquad \sec\theta = \frac{r}{a}$$

$$\tan\theta = \frac{b}{a} \qquad \cot\theta = \frac{a}{b}$$

### ÁNGULOS COTERMINALES

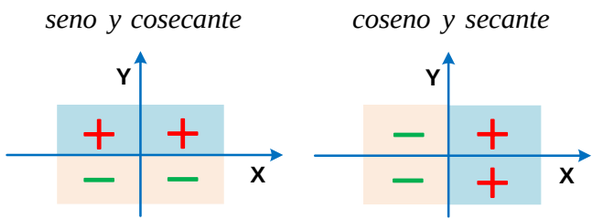



$\alpha - \beta = 360°n$ ; $n \in \mathbb{Z}$

α y β *coterminales*

$RT(\alpha) = RT(\beta)$

### SIGNOS DE LAS RT EN LOS CUADRANTES




| tangente y cotangente | IC | IIC | IIIC | IVC |
|---|---|---|---|---|
| | + | − | + | − |

### RT DE ÁNGULOS CUADRANTALES

$\theta$ *es cuadrantal* $\Leftrightarrow \theta = 90°n;\ n \in \mathbb{Z}$

| | 0° | 90° | 180° | 270° | 360° |
|---|---|---|---|---|---|
| ***sen*** | *0* | *1* | *0* | *–1* | *0* |
| ***cos*** | *1* | *0* | *–1* | *0* | *1* |
| ***tan*** | *0* | *ND* | *0* | *ND* | *0* |
| ***cot*** | *ND* | *0* | *ND* | *0* | *ND* |
| ***sec*** | *1* | *ND* | *–1* | *ND* | *1* |
| ***csc*** | *ND* | *1* | *ND* | *–1* | *ND* |

ND: no definido

## V. IDENTIDADES TRIGONOMÉTRICAS (IT)

### IT FUNDAMENTALES

| *IT RECÍPROCAS* | *IT PITAGÓRICAS* |
|---|---|
| $\operatorname{sen}\theta\csc\theta = 1$ | $\operatorname{sen}^2\theta + \cos^2\theta = 1$ |
| $\cos\theta\sec\theta = 1$ | $1 + \tan^2\theta = \sec^2\theta$ |
| $\tan\theta\cot\theta = 1$ | $1 + \cot^2\theta = \csc^2\theta$ |

*Tener en cuenta...*

1) $\sec\theta + \tan\theta = \dfrac{a}{b} \Leftrightarrow \sec\theta - \tan\theta = \dfrac{b}{a}$

2) $\csc\theta + \cot\theta = \dfrac{a}{b} \Leftrightarrow \csc\theta - \cot\theta = \dfrac{b}{a}$

### IDENTIDADES AUXILIARES

1) $\operatorname{sen}^4\theta + \cos^4\theta = 1 - 2\operatorname{sen}^2\theta\cos^2\theta$
2) $\operatorname{sen}^6\theta + \cos^6\theta = 1 - 3\operatorname{sen}^2\theta\cos^2\theta$
3) $\tan\theta + \cot\theta = \sec\theta\csc\theta$
4) $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$
5) $(1 + \operatorname{sen}\theta + \cos\theta)^2 = 2(1 + \operatorname{sen}\theta)(1 + \cos\theta)$
6) $(1 - \operatorname{sen}\theta - \cos\theta)^2 = 2(1 - \operatorname{sen}\theta)(1 - \cos\theta)$

*Tener en cuenta...Si*

$a\operatorname{sen}\theta + b\cos\theta = c$ *y* $a^2 + b^2 = c^2$ ⇒

$$\operatorname{sen}\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c}$$

## VI. IT DE ÁNGULOS COMPUESTOS

$$\operatorname{sen}(\theta+\alpha)=\operatorname{sen}\theta\cos\alpha+\cos\theta\operatorname{sen}\alpha$$
$$\operatorname{sen}(\theta-\alpha)=\operatorname{sen}\theta\cos\alpha-\cos\theta\operatorname{sen}\alpha$$

$$\cos(\theta+\alpha)=\cos\theta\cos\alpha-\operatorname{sen}\theta\operatorname{sen}\alpha$$
$$\cos(\theta-\alpha)=\cos\theta\cos\alpha+\operatorname{sen}\theta\operatorname{sen}\alpha$$

$$\tan(\alpha+\theta)=\frac{\tan\alpha+\tan\theta}{1-\tan\alpha.\tan\theta} \qquad \tan(\alpha-\theta)=\frac{\tan\alpha-\tan\theta}{1+\tan\alpha.\tan\theta}$$

*Tener en cuenta…*

$$\operatorname{sen}(\alpha+\theta)\operatorname{sen}(\alpha-\theta)=\operatorname{sen}^2\alpha-\operatorname{sen}^2\theta$$
$$\cos(\alpha+\theta)\cos(\alpha-\theta)=\cos^2\alpha-\operatorname{sen}^2\theta$$
$$\tan\alpha+\tan\theta=\frac{\operatorname{sen}(\alpha+\theta)}{\cos\alpha\cos\theta}$$
$$\tan\alpha-\tan\theta=\frac{\operatorname{sen}(\alpha-\theta)}{\cos\alpha\cos\theta}$$
$$\tan\alpha+\tan\theta+\tan\alpha.\tan\theta.\tan(\alpha+\theta)=\tan(\alpha+\theta)$$
$$\tan\alpha-\tan\theta-\tan\alpha.\tan\theta.\tan(\alpha-\theta)=\tan(\alpha-\theta)$$

**PROPIEDADES**

1) $Si\ A+B+C=180°\Rightarrow$
$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

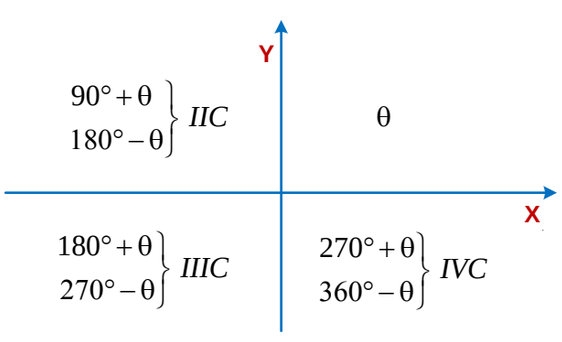
2) $Si\ A+B+C=90°\Rightarrow$
$$\cot A+\cot B+\cot C=\cot A\cot B\cot C$$

3) Sea $E=a\operatorname{sen}x+b\cos x;\ x\in\mathbb{R}\Rightarrow$
$$-\sqrt{a^2+b^2}\le E\le\sqrt{a^2+b^2}$$
Donde: $E_{\max}=\sqrt{a^2+b^2}\ \wedge\ E_{\min}=-\sqrt{a^2+b^2}$

## VII. REDUCCIÓN AL PRIMER CUADRANTE

**CASO I: PARA ÁNGULOS POSITIVOS MENORES A UNA VUELTA**

*Considerando a θ como un ángulo agudo.*




$$R.T.(90°+\theta)=(signo)\,coRT(\theta)$$
$$R.T.(180°\pm\theta)=(signo)\,R.T.(\theta)$$
$$R.T.(270°\pm\theta)=(signo)\,coRT(\theta)$$
$$R.T.(360°-\theta)=(signo)\,R.T.(\theta)$$

Donde el signo (+) o (–) del lado derecho de las igualdades depende del signo de la razón trigonométrica (R.T.) del ángulo que aparece a la izquierda.

*Tener en cuenta…*

| ***Razón trigonométrica (R.T.)*** | ***Co-razón trigonométrica (coRT)*** |
|---|---|
| *seno* | *coseno* |
| *coseno* | *seno* |
| *tangente* | *cotangente* |
| *cotangente* | *tangente* |
| *secante* | *cosecante* |
| *cosecante* | *secante* |

**CASO II: PARA ÁNGULOS POSITIVOS MAYORES A UNA VUELTA**

$$R.T.(360°n\pm\theta)=R.T.(\pm\theta)\ o$$
$$R.T.(2n\pi\pm\theta)=R.T.(\pm\theta)$$

Donde $n\in\mathbb{Z}$ indica el número entero de vueltas (360°) o (2π) que contiene el ángulo a reducir.

**CASO III: PARA ÁNGULOS NEGATIVOS**

$$\operatorname{sen}(-\theta)=-\operatorname{sen}\theta \qquad \csc(-\theta)=-\operatorname{cs}c\,\theta$$
$$\cos(-\theta)=\cos\theta \qquad \sec(-\theta)=\sec\theta$$
$$\tan(-\theta)=-\tan\theta \qquad \cot(-\theta)=-\cot\theta$$

**PROPIEDADES**

1) $Si\ A+B=180°\Rightarrow\begin{cases}\operatorname{sen}A=\operatorname{sen}B; & \csc A=\csc B\\ \cos A=-\cos B; & \sec A=-\sec B\\ \tan A=-\tan B; & \cot A=-\cot B\end{cases}$

2) $Si\ A+B=270°\Rightarrow\begin{cases}\operatorname{sen}A=-\cos B; & \cos A=-\operatorname{sen}B\\ \tan A=\cot B; & \cot A=\tan B\\ \sec A=-\csc B; & \csc A=-\sec B\end{cases}$

3) $Si\ A+B=360°\Rightarrow\begin{cases}\operatorname{sen}A=-\operatorname{sen}B; & \csc A=-\csc B\\ \cos A=\cos B; & \sec A=\sec B\\ \tan A=-\tan B; & \cot A=-\cot B\end{cases}$

## VIII. IT DE ÁNGULOS MULTIPLES

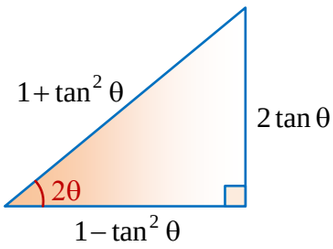
### IT DEL ÁNGULO DOBLE

$$\operatorname{sen}2\theta = 2\operatorname{sen}\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \operatorname{sen}^2\theta \Leftrightarrow \begin{cases} \cos 2\theta = 2\cos^2\theta - 1 \\ \cos 2\theta = 1 - 2\operatorname{sen}^2\theta \end{cases}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

*Tener en cuenta...*




$$\operatorname{sen}2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

### IDENTIDADES AUXILIARES

1) $2\operatorname{sen}^2\theta = 1 - \cos 2\theta$
2) $2\cos^2\theta = 1 + \cos 2\theta$
3) $\cot\theta + \tan\theta = 2\csc 2\theta$
4) $\cot\theta - \tan\theta = 2\cot 2\theta$
5) $(\operatorname{sen}\theta + \cos\theta)^2 = 1 + \operatorname{sen}2\theta$
6) $\operatorname{sen}^4\theta + \cos^4\theta = \frac{3}{4} + \frac{3}{4}\cos 4\theta$
7) $\operatorname{sen}^6\theta + \cos^6\theta = \frac{5}{8} + \frac{3}{8}\cos 4\theta$

### IT DEL ÁNGULO MITAD

$$\operatorname{sen}\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

El signo (+) o (-) depende del cuadrante al cual pertenece $\frac{\theta}{2}$

*Tener en cuenta...*

1) $\cot\frac{\theta}{2} = \csc\theta + \cot\theta$
2) $\tan\frac{\theta}{2} = \csc\theta - \cot\theta$
3) $\cot\frac{\theta}{2} - \tan\frac{\theta}{2} = 2\cot\theta$
4) $\cot\frac{\theta}{2} + \tan\frac{\theta}{2} = 2\csc\theta$

### IT DEL ÁNGULO TRIPLE

$$\operatorname{sen}3\theta = 3\operatorname{sen}\theta - 4\operatorname{sen}^3\theta \;\; o \;\; \operatorname{sen}3\theta = \operatorname{sen}\theta(2\cos 2\theta + 1)$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \;\; o \;\; \cos 3\theta = \cos\theta(2\cos 2\theta - 1)$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

*Tener en cuenta...*

1) $4\operatorname{sen}(60° - \theta)\operatorname{sen}\theta\operatorname{sen}(60° + \theta) = \operatorname{sen}3\theta$
2) $4\cos(60° - \theta)\cos\theta\cos(60° + \theta) = \cos 3\theta$
3) $\tan(60° - \theta)\tan\theta\tan(60° + \theta) = \tan 3\theta$
4) $\frac{\tan 3\theta}{\tan\theta} = \frac{2\cos 2\theta + 1}{2\cos 2\theta - 1}$

## IX. TRANSFORMACIONES TRIGONOMÉTRICAS

### DE (+) O (-) DE SENOS A PRODUCTO

$$\operatorname{sen}A + \operatorname{sen}B = 2\operatorname{sen}\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\operatorname{sen}A - \operatorname{sen}B = 2\cos\left(\frac{A+B}{2}\right)\operatorname{sen}\left(\frac{A-B}{2}\right)$$

### DE (+) O (-) DE COSENOS A PRODUCTO

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\operatorname{sen}\left(\frac{A+B}{2}\right)\operatorname{sen}\left(\frac{A-B}{2}\right)$$

### DE PRODUCTO A SUMA O DIFERENCIA

$$2sen x\cos y = sen(x+y) + sen(x-y)$$

$$2\cos x\cos y = \cos(x+y) + \cos(x-y)$$

$$2sen x\, seny = \cos(x-y) - \cos(x+y)$$

### PROPIEDADES

1) *Si* $A + B + C = 180° \Rightarrow$

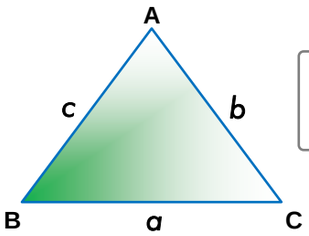
$$\begin{cases} \operatorname{sen}A + \operatorname{sen}B + \operatorname{sen}C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \\ \cos A + \cos B + \cos C = 4sen\frac{A}{2}sen\frac{B}{2}sen\frac{C}{2} + 1 \end{cases}$$

2) *Si* $A + B + C = 360° \Rightarrow$

$$\begin{cases} \operatorname{sen}A + \operatorname{sen}B + \operatorname{sen}C = 4sen\frac{A}{2}sen\frac{B}{2}sen\frac{C}{2} \\ \cos A + \cos B + \cos C = -4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 1 \end{cases}$$

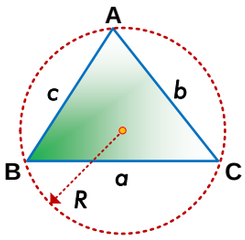
## X. TRIÁNGULOS OBLICUÁNGULOS

### TEOREMA DE SENOS (O LEY DE SENOS)




$$\frac{a}{\text{sen}} \quad \frac{b}{\text{sen}} \quad \frac{c}{\text{sen}}$$

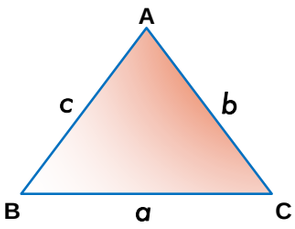
### *Con el circunradio*




$$\frac{a}{\text{sen}A}=\frac{b}{\text{sen}B}=\frac{c}{\text{sen}C}=2R$$

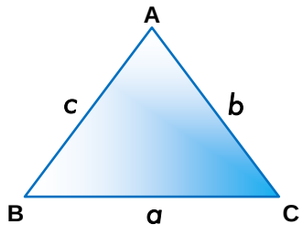
$$\begin{aligned} a &= 2R\text{sen}A \\ b &= 2R\text{sen}B \\ c &= 2R\text{sen}C \end{aligned}$$

### TEOREMA DE COSENOS (O LEY DE COSENOS)




$$\begin{aligned} a^2 &= b^2+c^2-2bc\cos A \\ b^2 &= a^2+c^2-2ac\cos B \\ c^2 &= a^2+b^2-2ab\cos C \end{aligned}$$

### TEOREMA DE TANGENTES (O LEY DE TANGENTES)

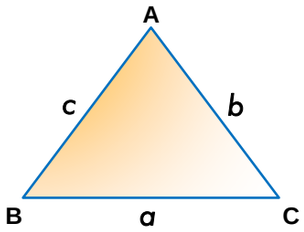



$$\frac{a-b}{a+b}=\frac{\tan\dfrac{A-B}{2}}{\tan\dfrac{A+B}{2}}$$

$$\frac{a-c}{a+c}=\frac{\tan\dfrac{A-C}{2}}{\tan\dfrac{A+C}{2}}$$

$$\frac{b-c}{b+c}=\frac{\tan\dfrac{B-C}{2}}{\tan\dfrac{B+C}{2}}$$

### TEOREMA DE PROYECCIONES




$$\begin{array}{lll} a & b\cos C & c\cos B \\ b & a\cos C & c\cos A \\ c & a\cos B & b\cos A \end{array}$$